The Computational Complexity of two Card Games with Theoretical Applications

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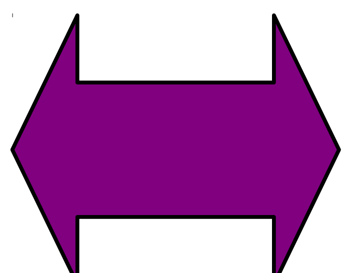
Hungarian Academy of Sciences

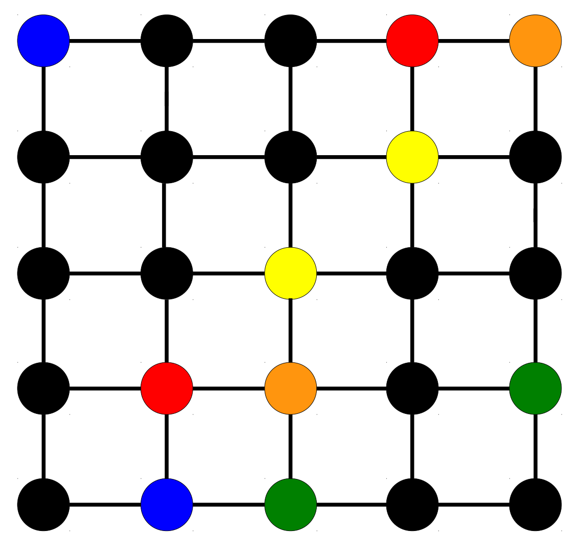
We study...

...games and puzzles which can be naturally re-formulated as variations of well-known graph problems.

Flow



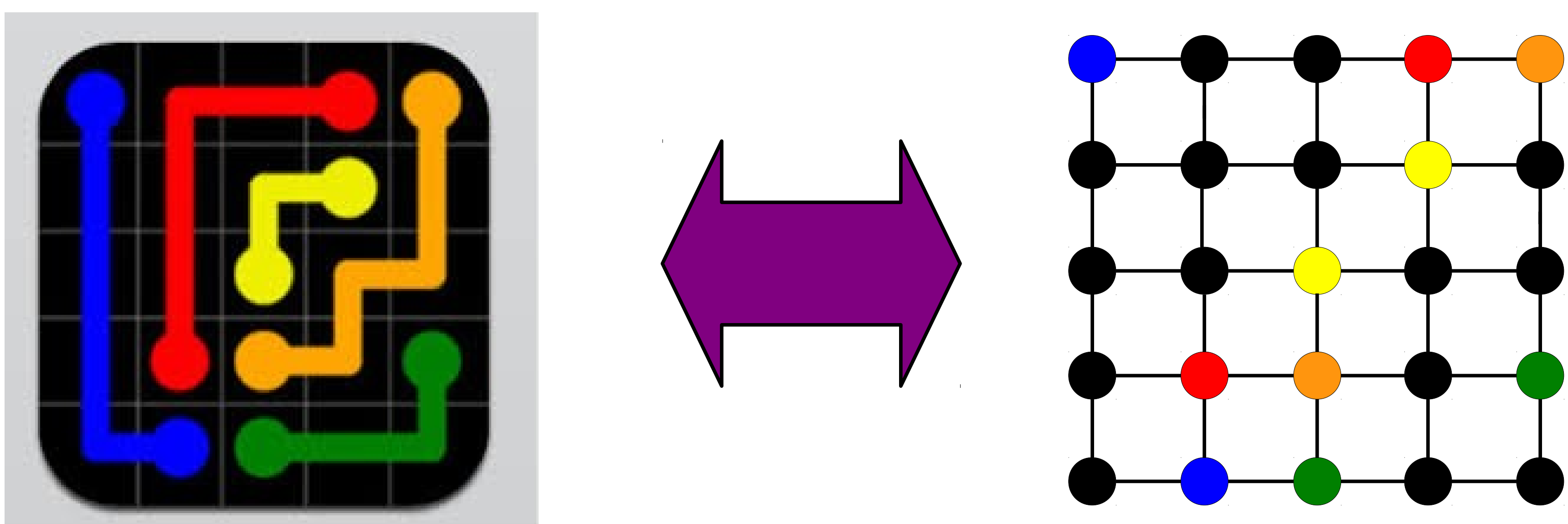
Vertex-disjoint paths



Who cares?

- Borrow known complexity results regarding the problem to prove the complexity of the game.
- Use the intuition provided by the game to advance knowledge about the problem.

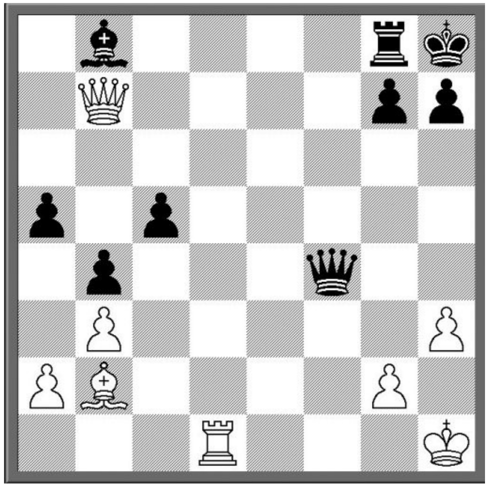
Flow **Vertex-disjoint paths**



Parameterized Games

- Games and Puzzles usually have many realistic parameters expected to take moderate values.
  - Distinguish between *truly* hard games and *parameterized-efficient* games.

- Study short games.



It is White to move and checkmate in 3 moves.

Today

- Multidimentional Matching
- Set Packing
- Edge Dominating Set

- (Edge) Hamiltonian Path

The Computational Complexity of the Game of

SET

Joint work with Michael Lampis
(Université Paris Dauphine)

The Game of Set - Rules

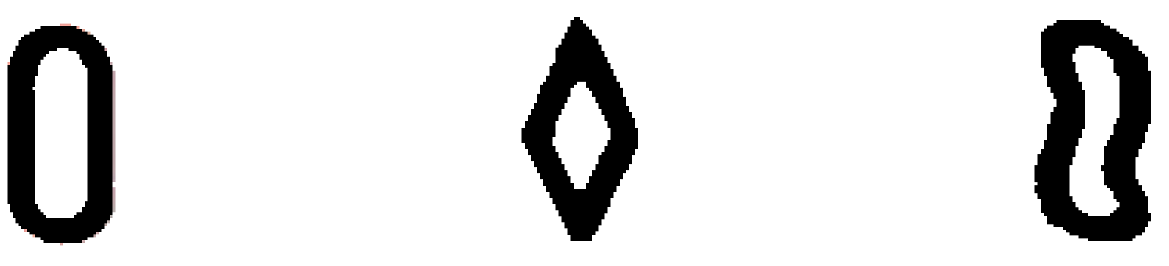
Each card has 4 attributes:

- Symbol
- Shading
- Color
- Number

The Game of Set - Rules

Each attribute can take one of 3 values:

- Symbol
  - Oval
  - Diamond
  - Squiggle



The Game of Set - Rules

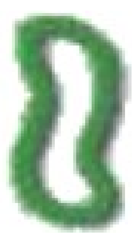
Each attribute can take one of 3 values:

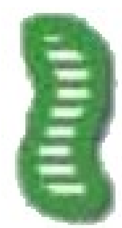
- Color
  - Red
  - Green
  - Purple

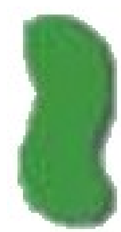
The Game of Set - Rules

Each attribute can take one of 3 values:

- Shading
  - Blank
  - Stripped
  - Solid



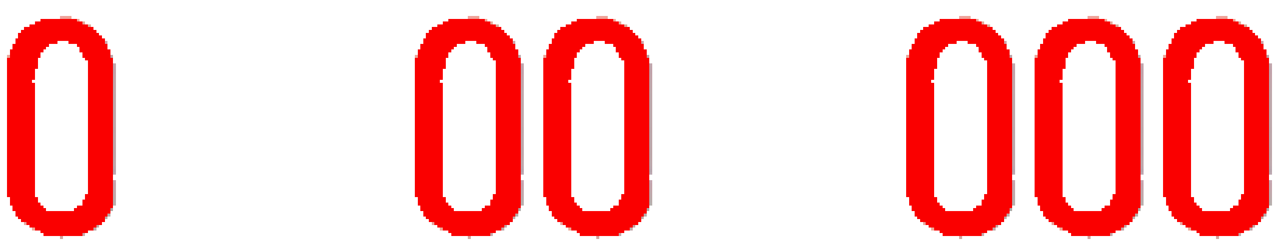




The Game of Set - Rules

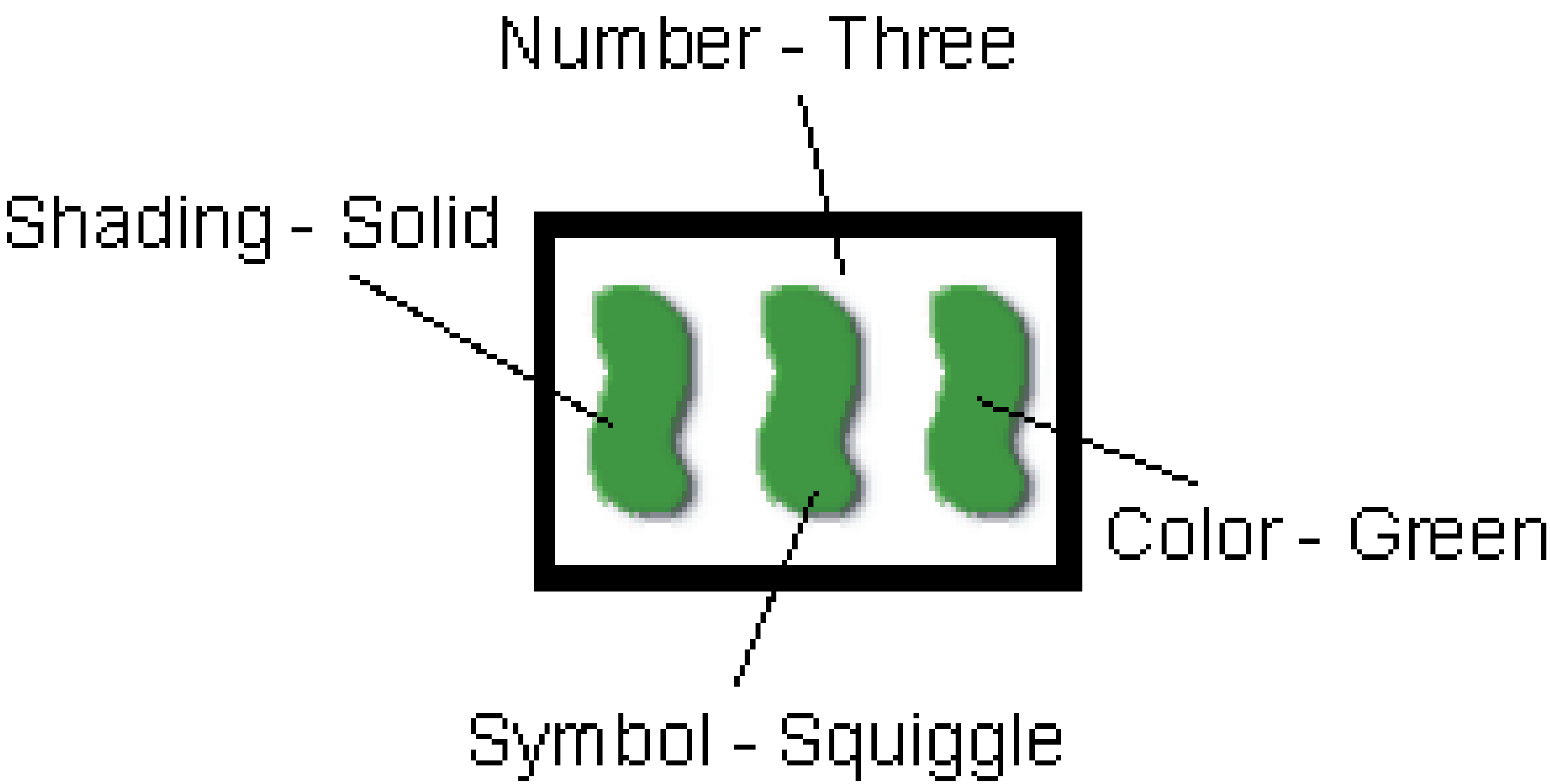
Each attribute can take one of 3 values:

- Number
  - One
  - Two
  - Three



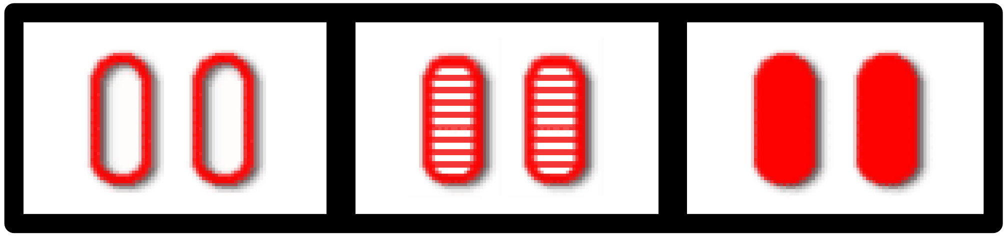
The Game of Set - Rules

There are $3^4 = 81$ different cards in total (one for each combination of values).



The Game of Set - Rules

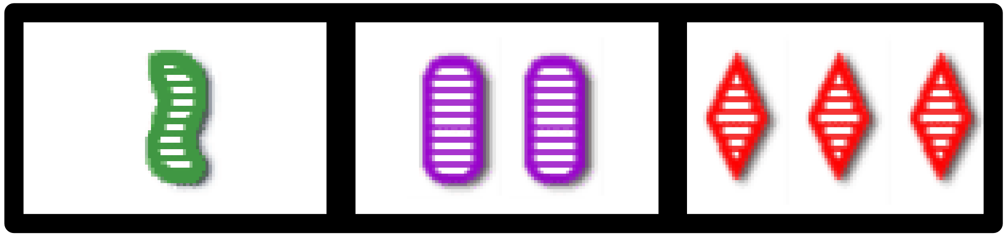
Valid set: 3 cards with values for each attribute being either *all the same* or *all different.*



- ✓ All have **same** color;
- ✓ all have **same** symbol;
- ✓ all have **same** number;
- ✓ all have **different** shadings.

The Game of Set - Rules

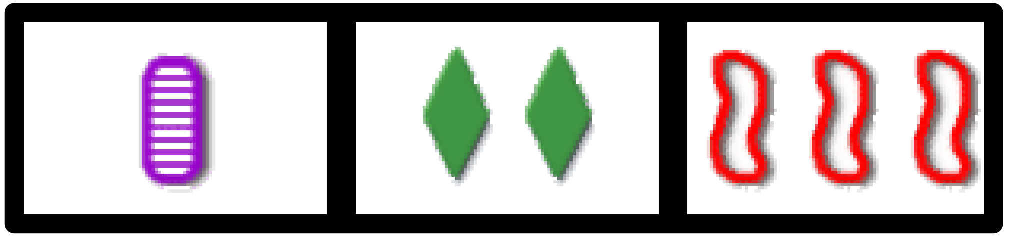
Valid set: 3 cards with values for each attribute being either *all the same* or *all different.*



- All have **different** colors;
- all have **different** symbols;
- all have **different** numbers;
- all have **same** shading.

The Game of Set - Rules

***Valid set**:* 3 cards with values for each attribute being either *all the same* or *all different.*



- ✓ All have **different** colors;
- ✓ all have **different** symbols;
- ✓ all have **different** number;
- ✓ all have **different** shadings.

The Game of Set - Rules

This is not a valid set!

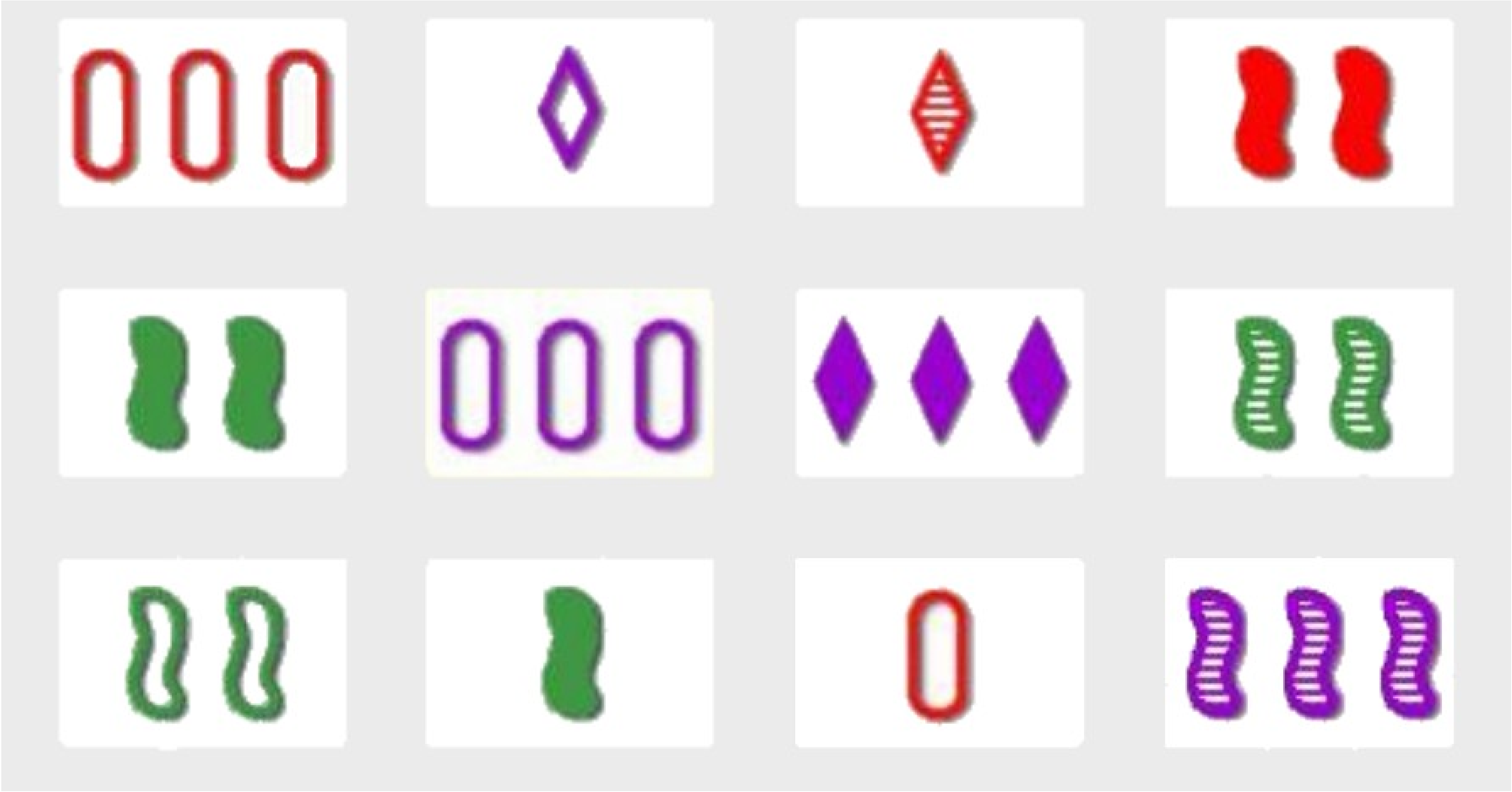
- ✓ All have **same** colors;
- ✓ all have **different** symbols;
- × **only 2/3** have same number;
- ✓ all have **different** shadings.

The Game of Set - Rules

- Deal 12 cards;
- Find a ***valid set.***

Naive way to find a valid set

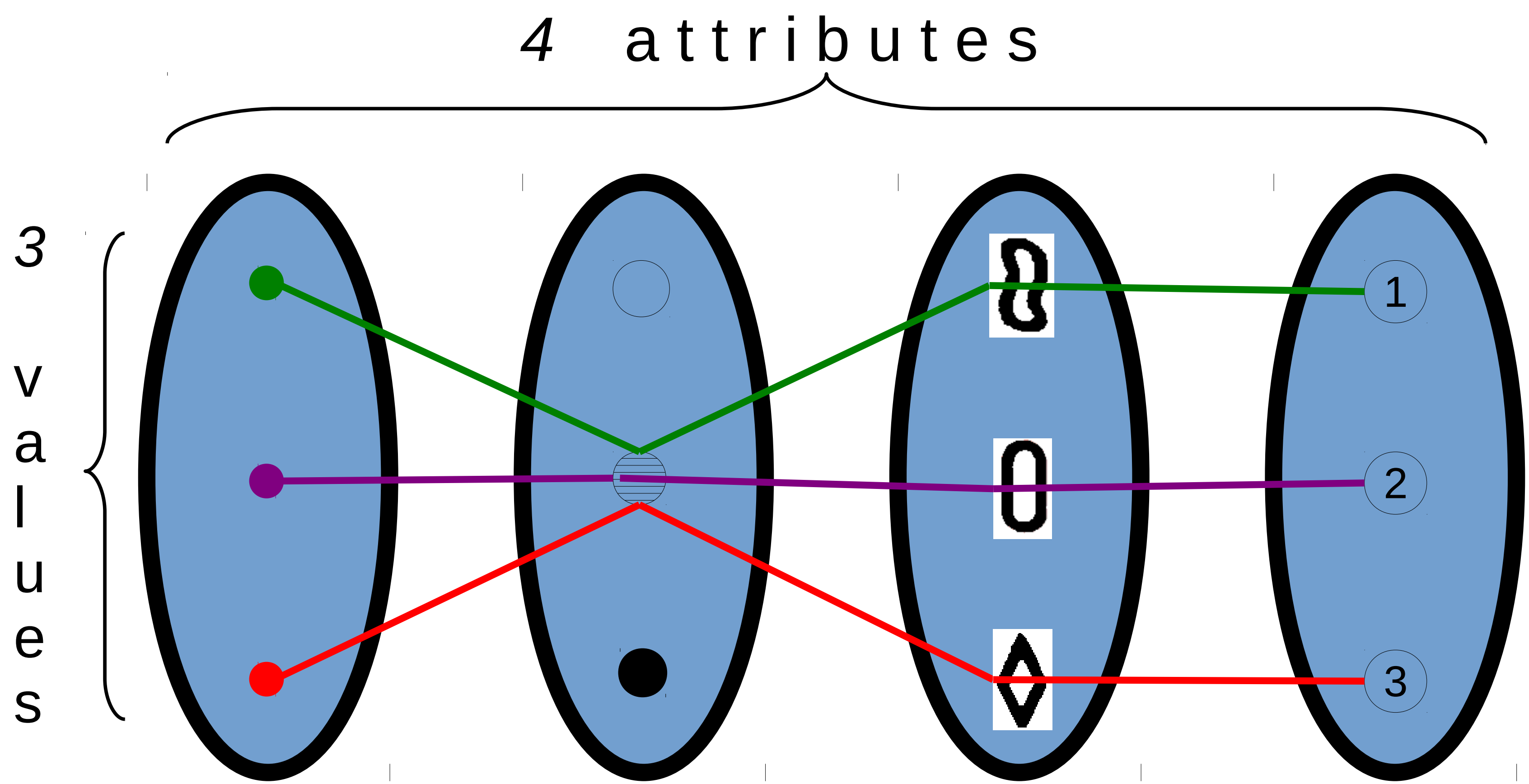
- Search among all possible triples.

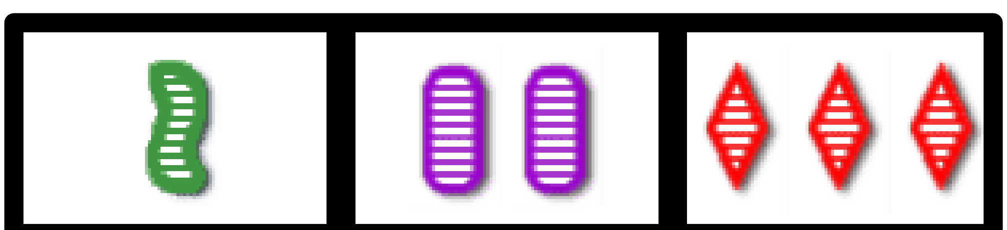


Generalization: k-1SET

- Input: m cards, n attributes, k values
- Question: Does there exist a valid set of k cards with all values the same or all values different?
- In the original game $m=12$, $n=4$ *and* $k=3$.

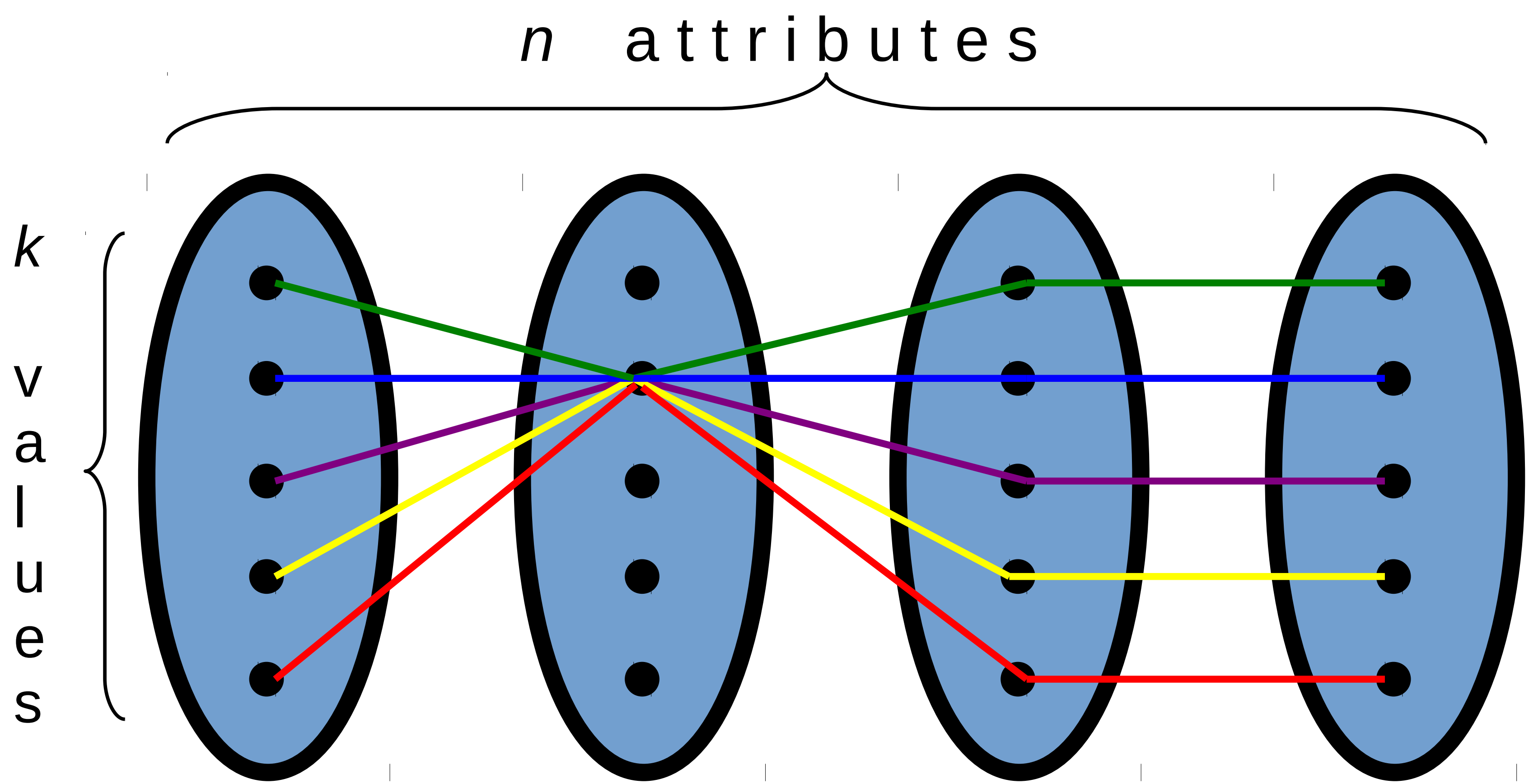
Hypergraph formulation





Cards = Hyper-edges
Attributes = Dimensions
# of Values = size of Parts

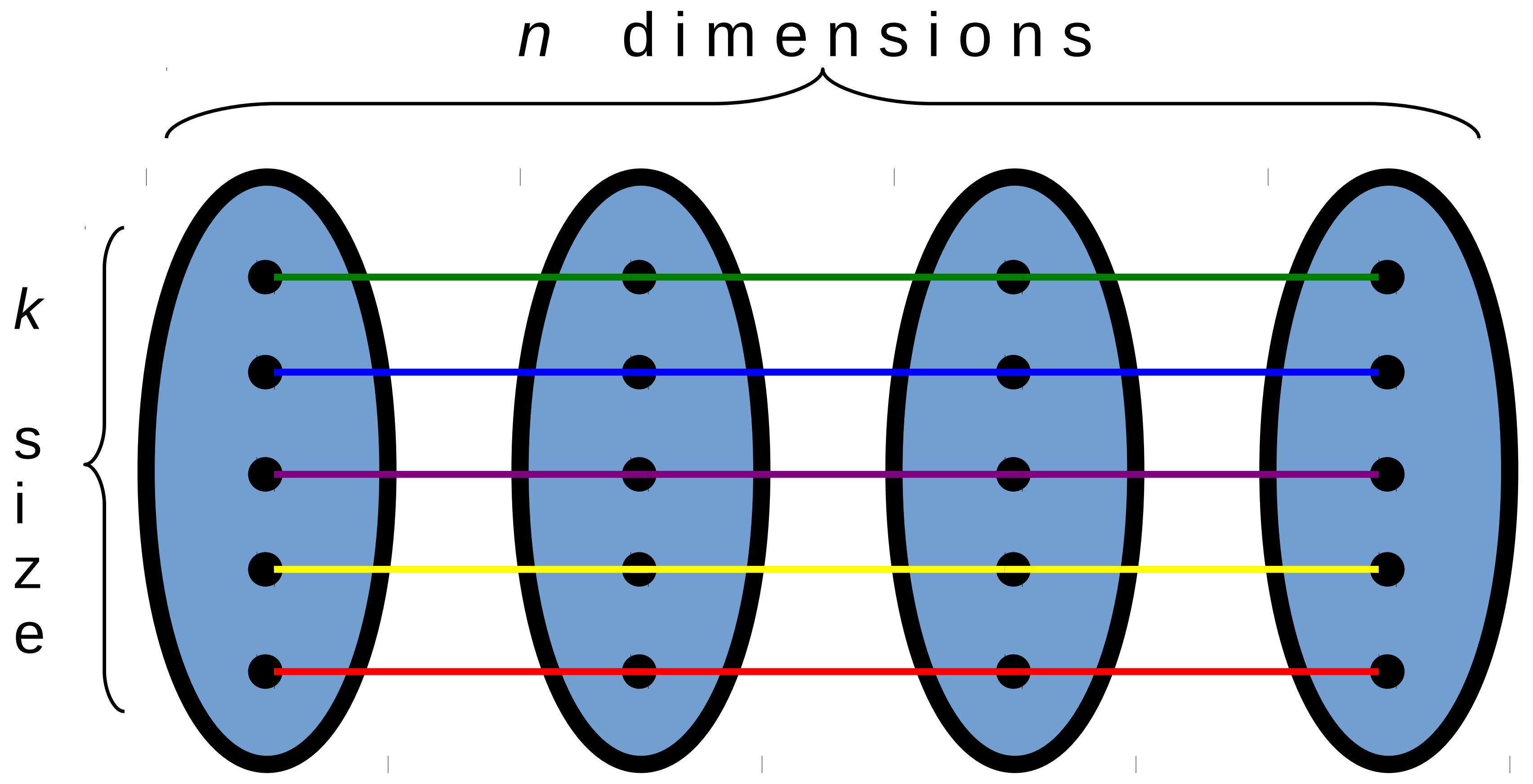
Hypergraph formulation



In SET, hyperedges are allowed to overlap as long as they all overlap on the same value.

Cards = Hyper-edges
Attributes = Dimensions
# of Values = size of Parts

Connection with *n*-Dim. Matching



<u>Perfect *n*-Dimensional Matching:</u>

Given a hypergraph H(V,E), pick k hyperedges such that all vertices are covered exactly once.

Formulation difficulties

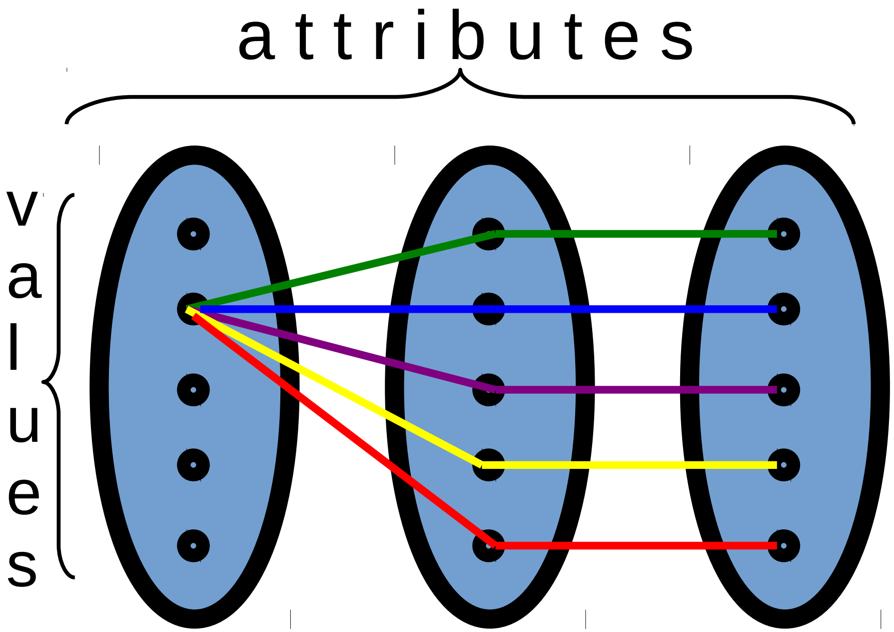
<u>Contradictory goals</u>:

- Define unbounded generalizations.
- Parameters m, n, k correspond to small integers.

Study parameterized complexity of the game (some of the above parameters are considered much smaller than others).

Complexity Results for *k*-1SET

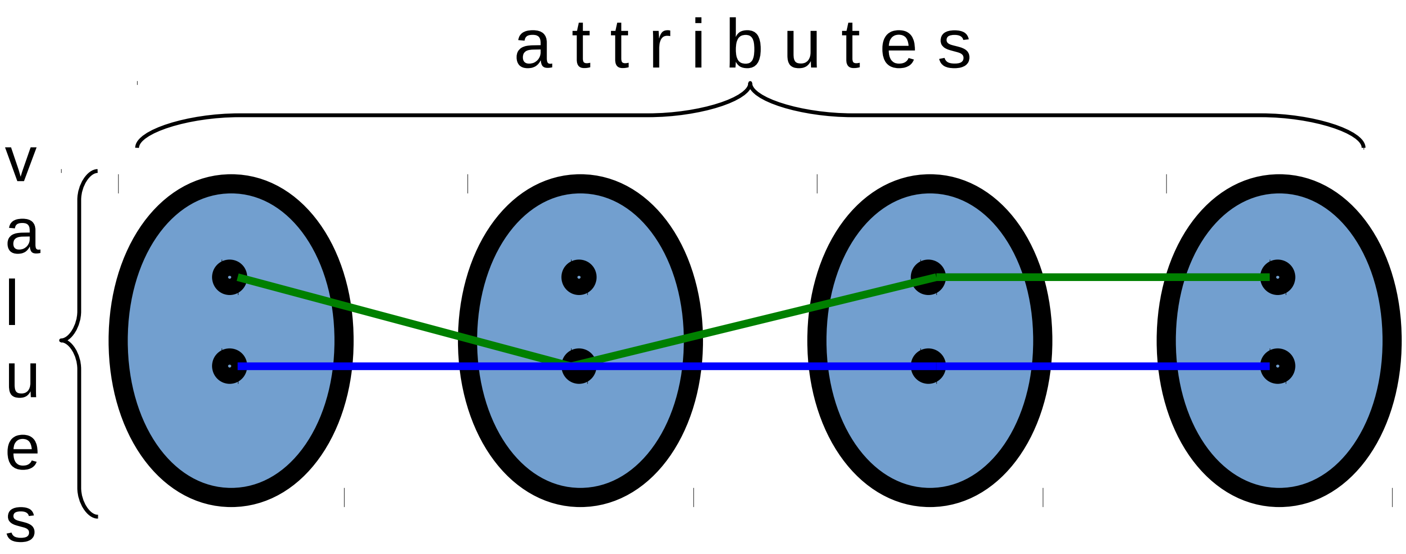
- For *m, k* unbounded:
  - $n = 2 \rightarrow$ P[1] (find a star or a bipartite matching)
  - $n \geq 3 \rightarrow$ NP-Complete[1]



1. Chaudhuri et al 2003.

Complexity Results for *k*-1SET

- For *m, n* unbounded:
  - $k = 2 \rightarrow$ trivial
  - k parameter $\rightarrow \binom{m}{k}$ (XP)



Complexity Results for *k*-1SET

- For *m*, *n* unbounded:
  - $k = 2 \rightarrow$ trivial
  - k parameter $\rightarrow \binom{m}{k}$ (XP)

<u>Our results:</u>

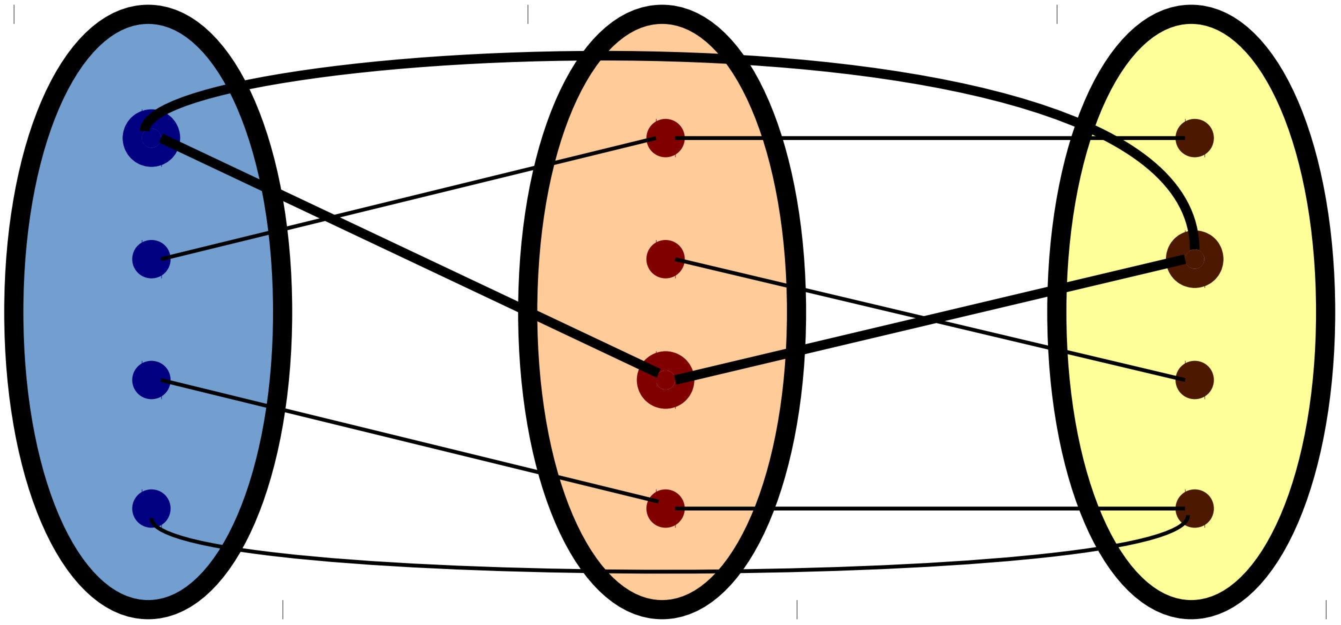
➡ *k-1SET parameterized by k is W-hard*

➡ *perfect n-DM parameterized by k is W-hard*

Reduction

From *k*-Multicolored Clique

- Input: *k*-partite graph, each part of size *n*
- Question: Does there exist a clique of size *k*?
- Parameter: *k*



Reduction

From k-Multicolored Clique

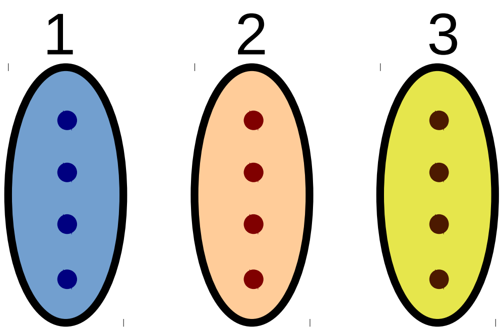
- Input: k-partite graph, each part of size n
- Question: Does there exist a clique of size k?
- Parameter: k

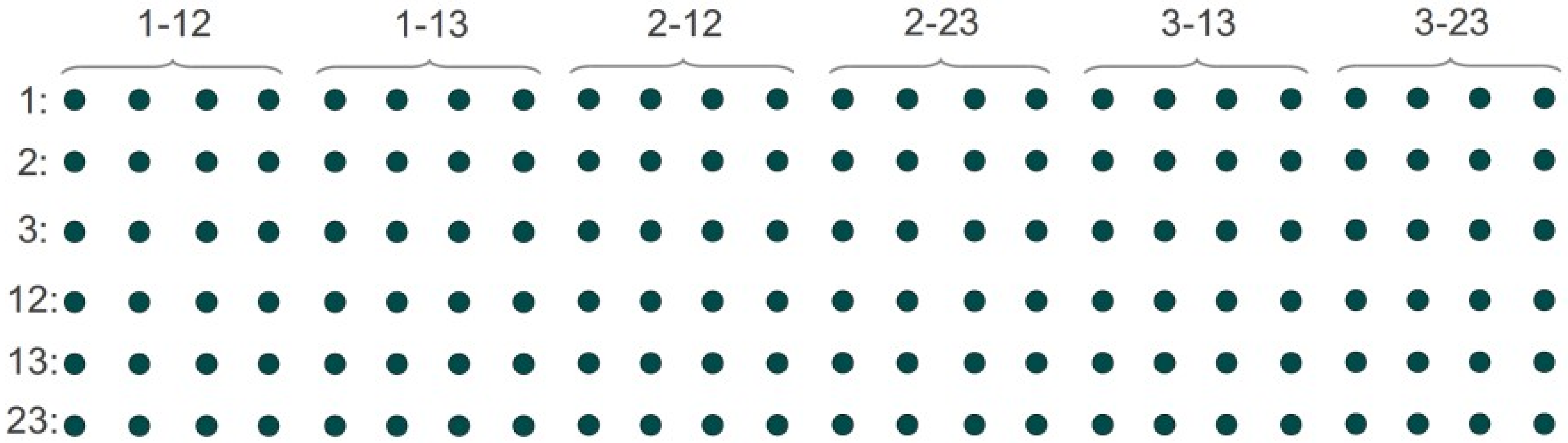
k-Multicolored Clique is W[1]-hard

Construction

The constructed multigraph:

- $n \cdot k(k\text{-}1)$ dimensions, in groups of size n
- $k+\binom{k}{2}$ possible different values



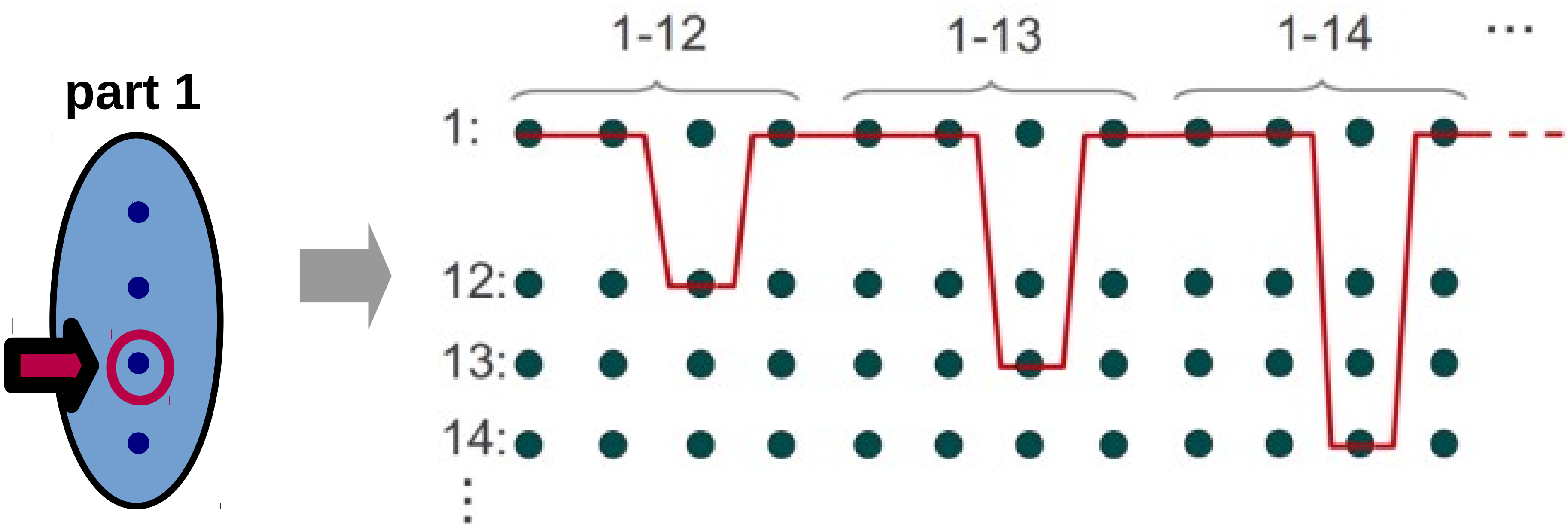


Construction

The constructed multigraph:

- For each vertex we construct a v-hyperedge.

  Example shows construction for 3rd vertex from part 1.

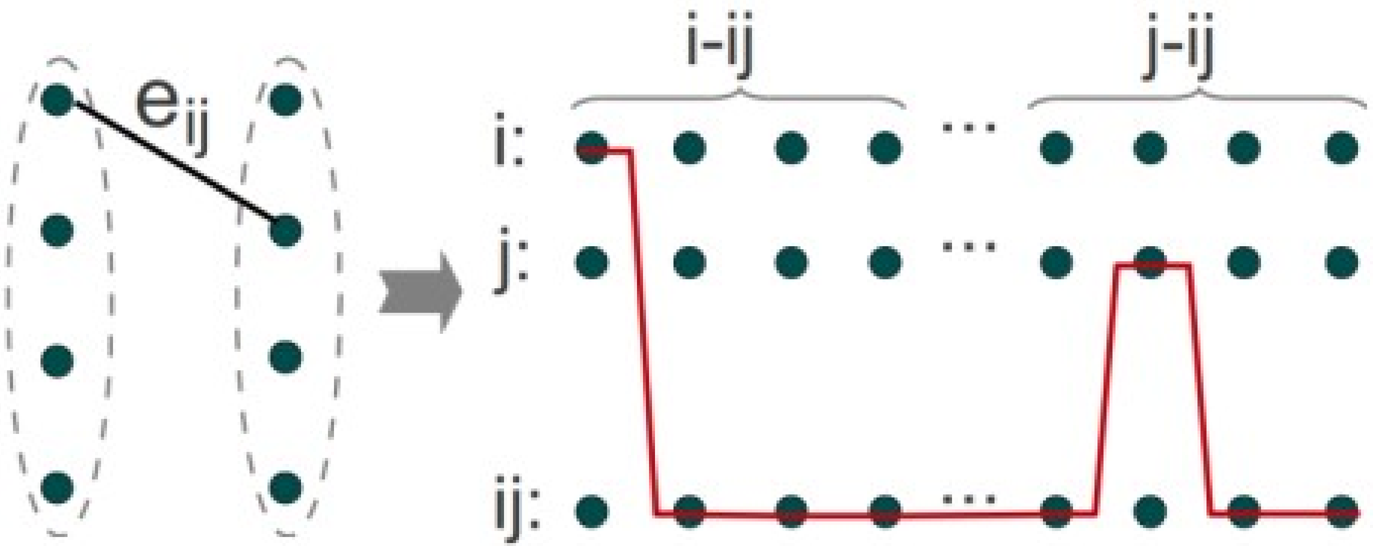


Construction

The constructed multigraph:

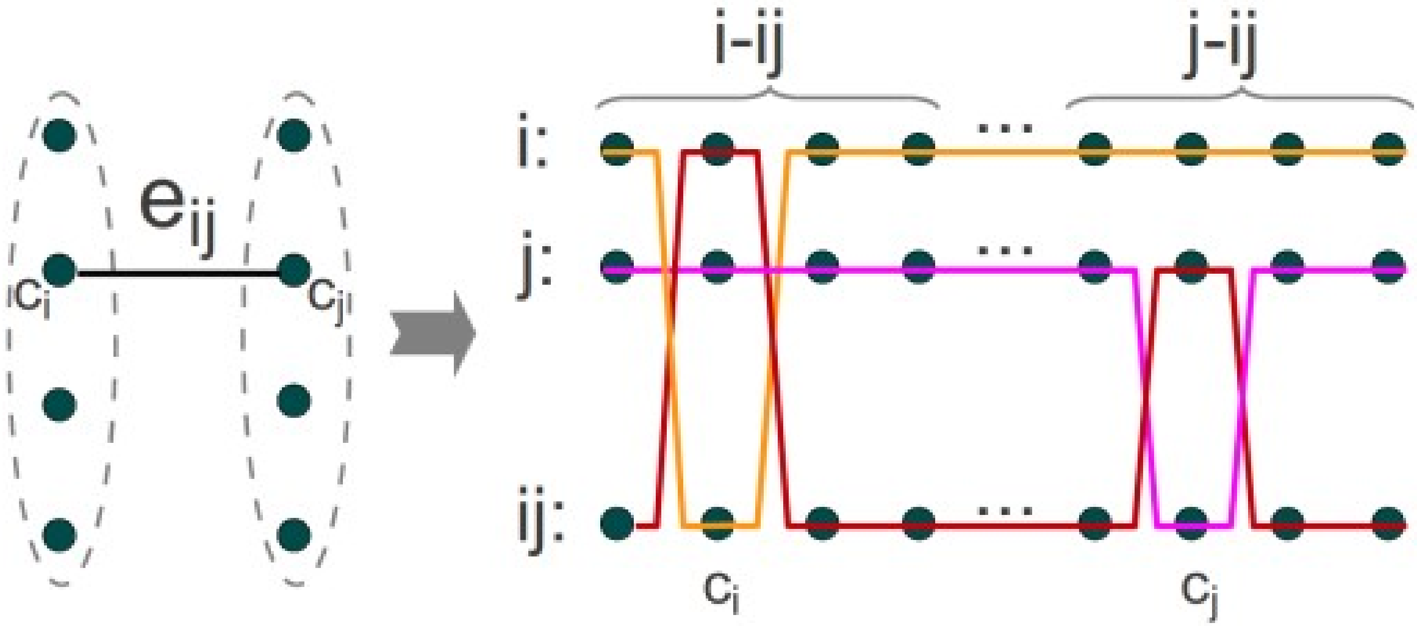
- For each edge we construct an e-hyperedge.

  Example shows edge connecting 1st vertex of part i with 2nd vertex of part j.

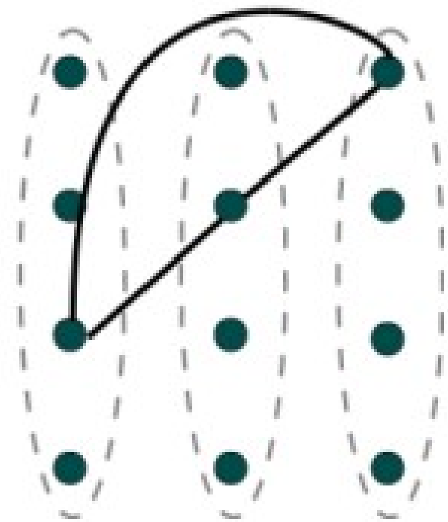


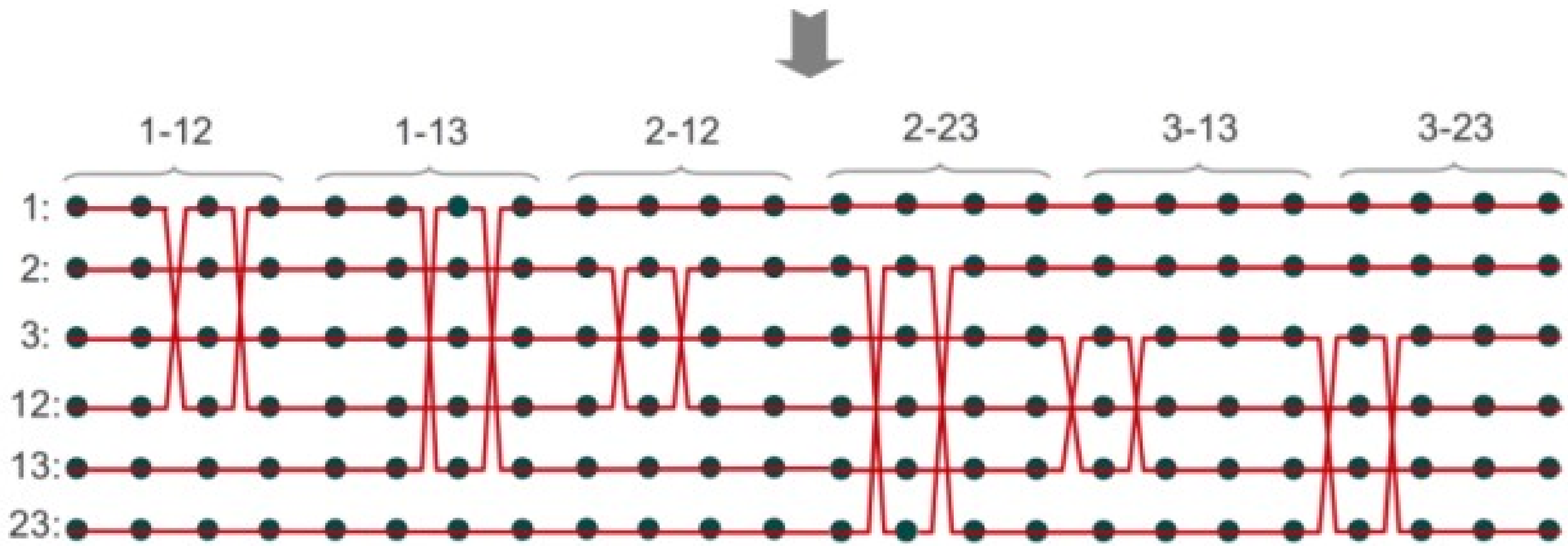
Correctness

- For each edge e_{ij} between parts i and j, the 3 hyperedges corresponding to i, j and ij cover the respective values entirely.



Example





Corollary for *k*-1SET

- Constructed hyperedges cannot all overlap, unless they correspond to the same parts.
- If there exists a valid set, it is also a perfect matching (and vice versa).

k-1SET parameterized by k is W-hard

Multi-round variations

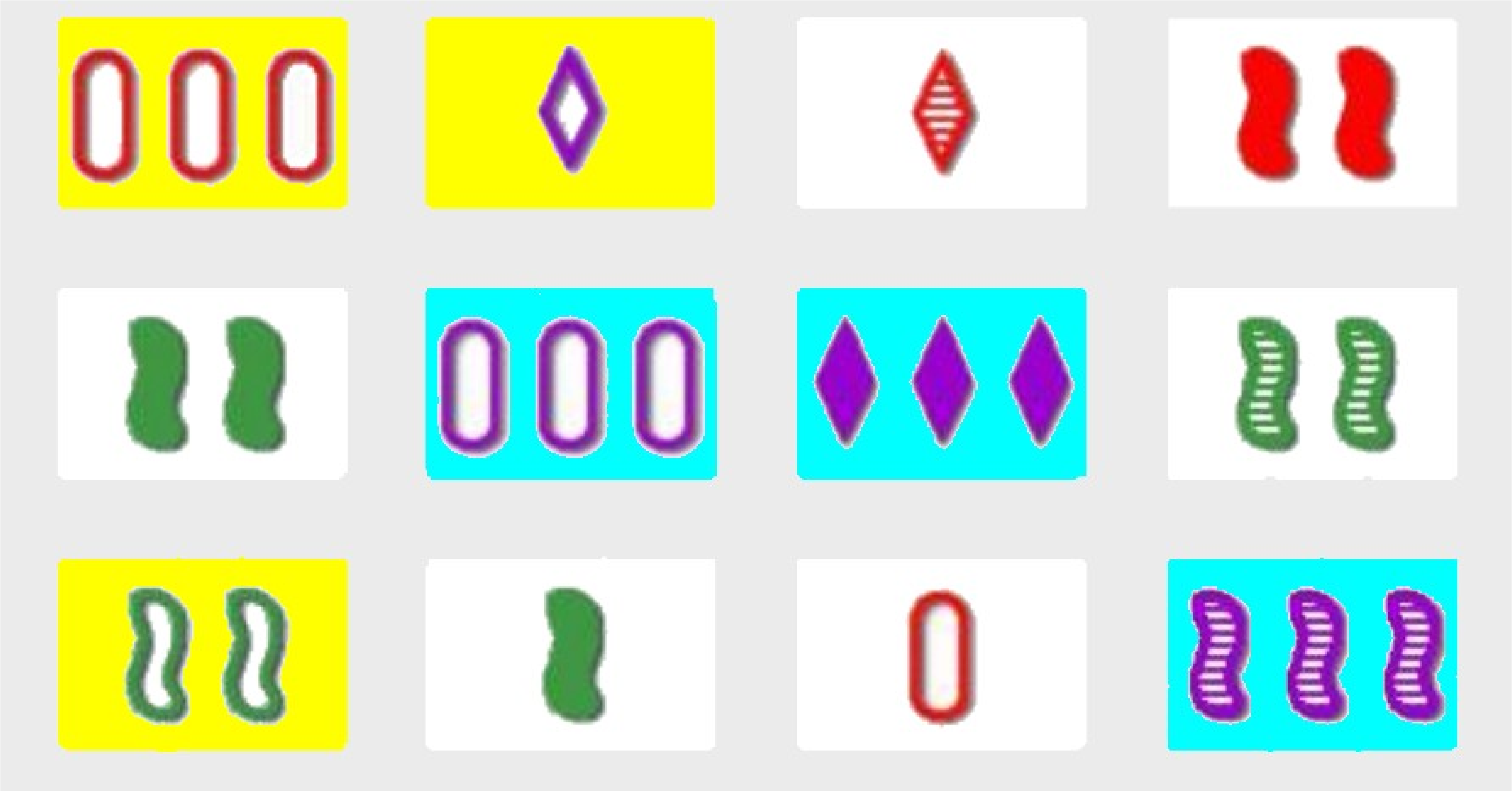
- Naive algorithm works for complete enumeration of all co-existing valid sets -without card removal.

Multi-round variations

- Alternative questions (card removal):
  - Max number of disjoint valid sets?

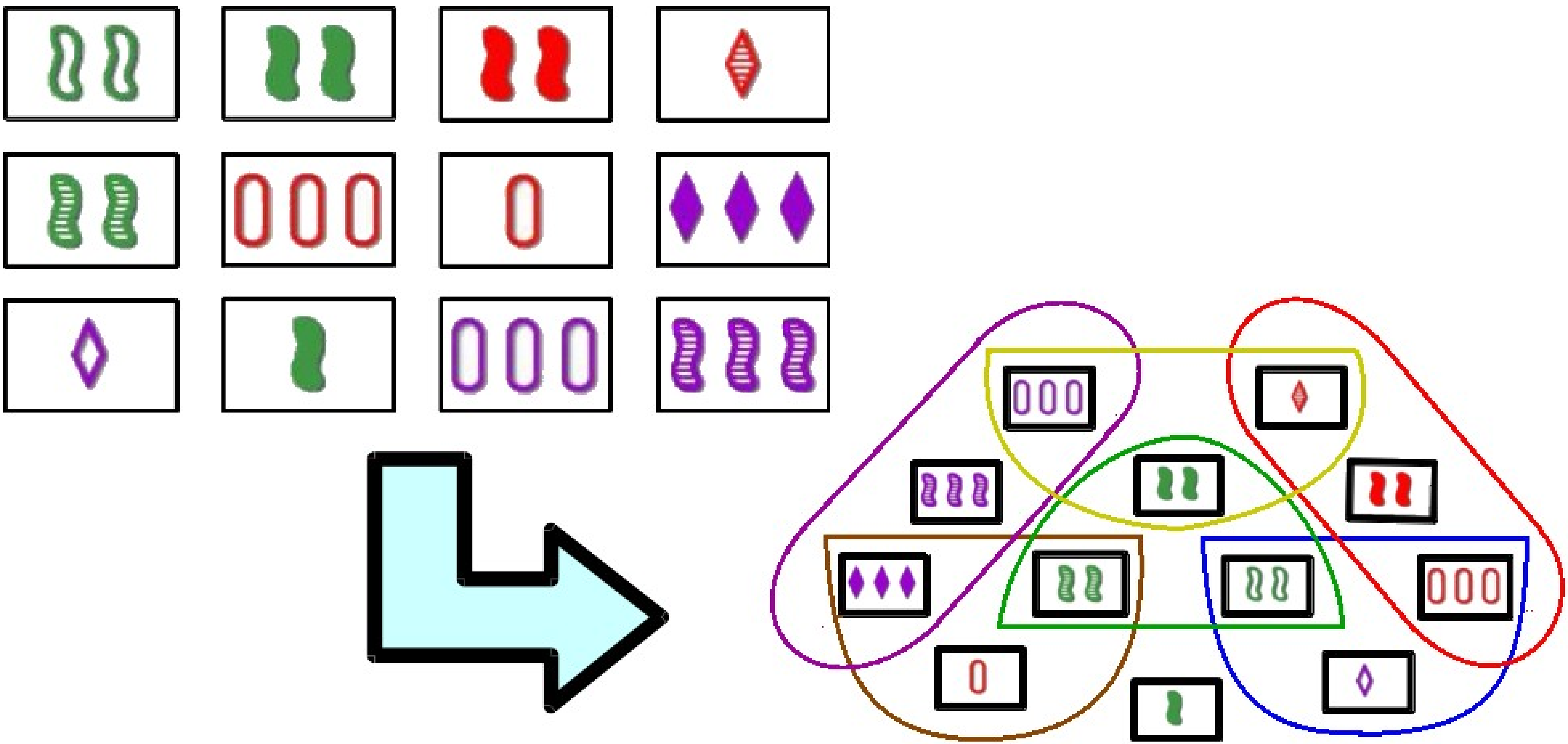
Multi-round variations

- Alternative questions (card removal):
  - Min number of valid sets that destroy all others?



Multi-round variations as hypergraph problems

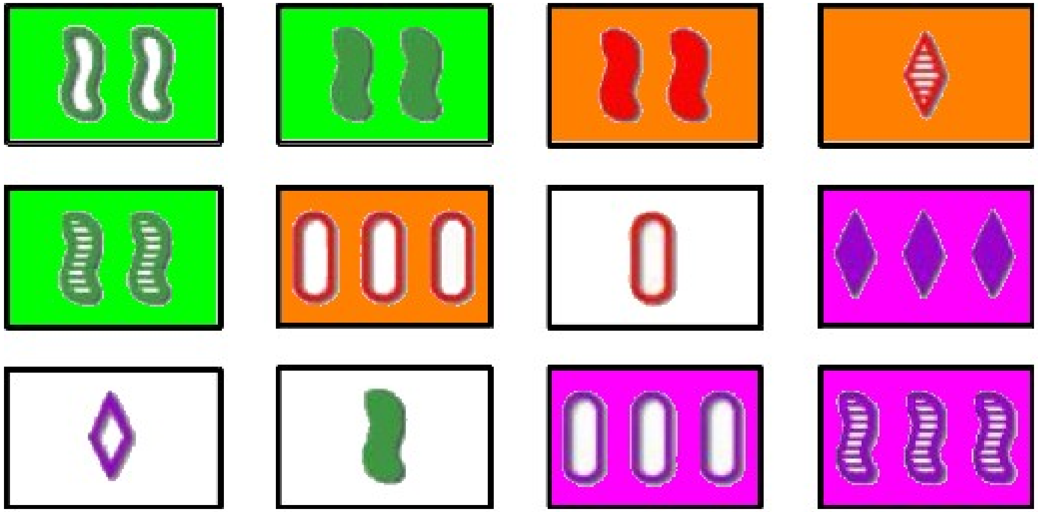
Construct a 3-uniform hypergraph:
vertices ↔ cards, hyperedges ↔ valid sets.

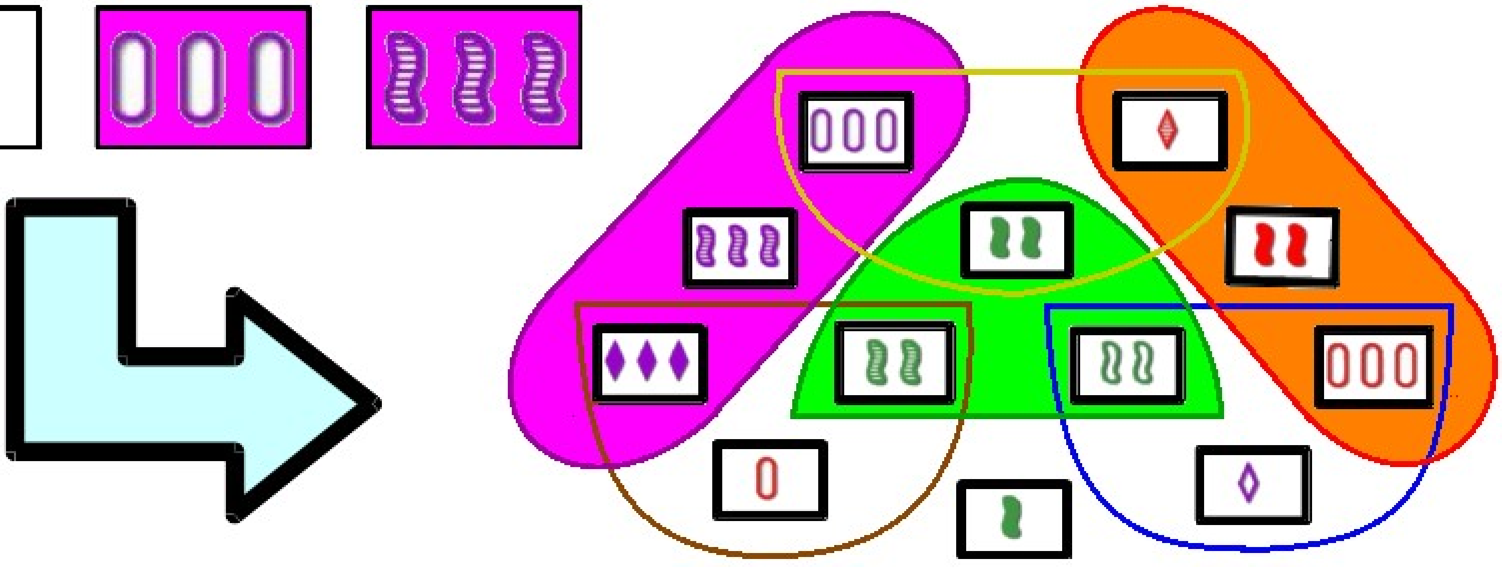


Max 3-*r*Set

Problem parameters: *m,n* unbounded, *k=3.*

Question: *Do there exist (at least) r disjoint valid sets?*





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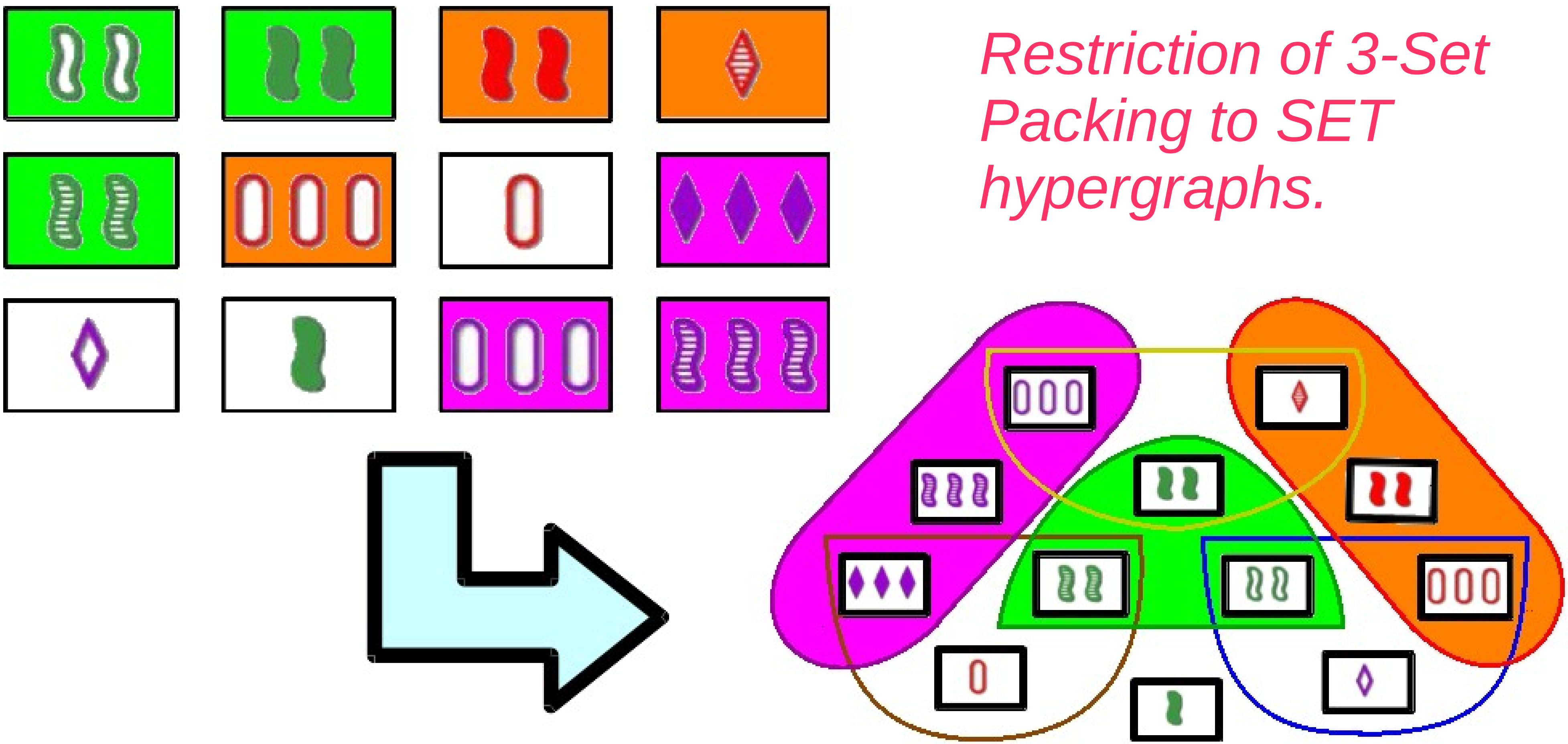
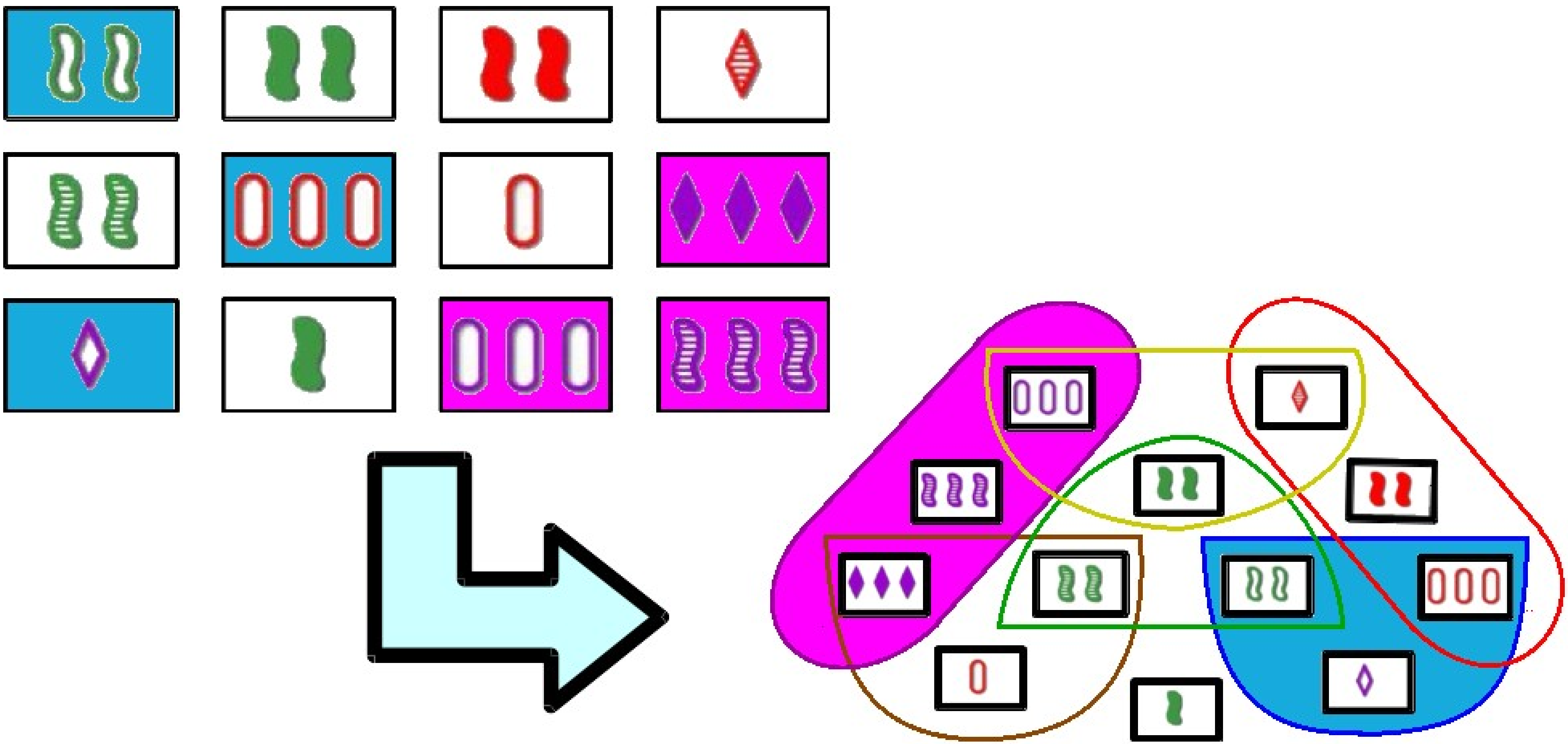


Restriction of 3-Set Packing to SET hypergraphs.

Min 3-*r*Set

<u>Problem parameters</u>: m,n unbounded, *k=3.*

<u>Question</u>: Do there exist (at most) r disjoint valid sets that overlap with all others?



Min 3-*r*Set

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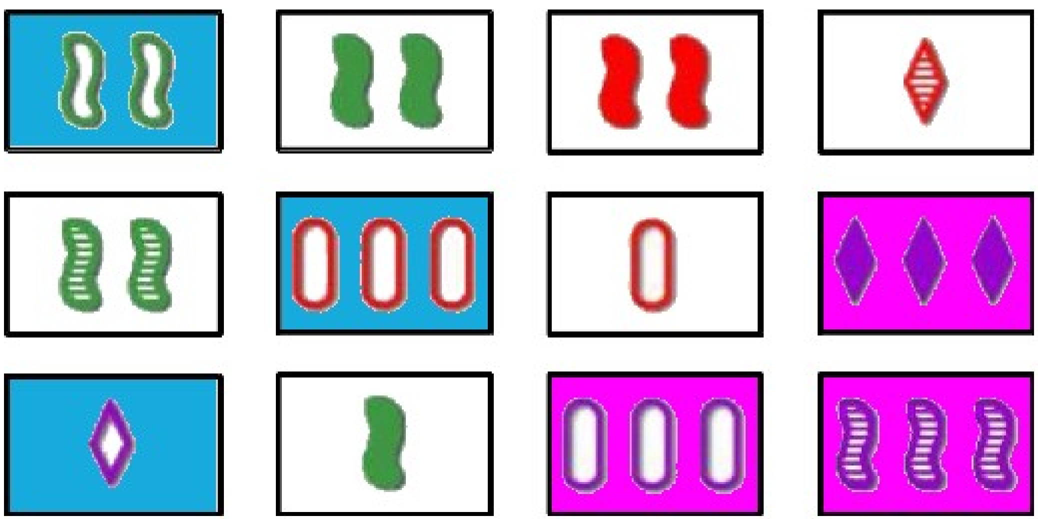
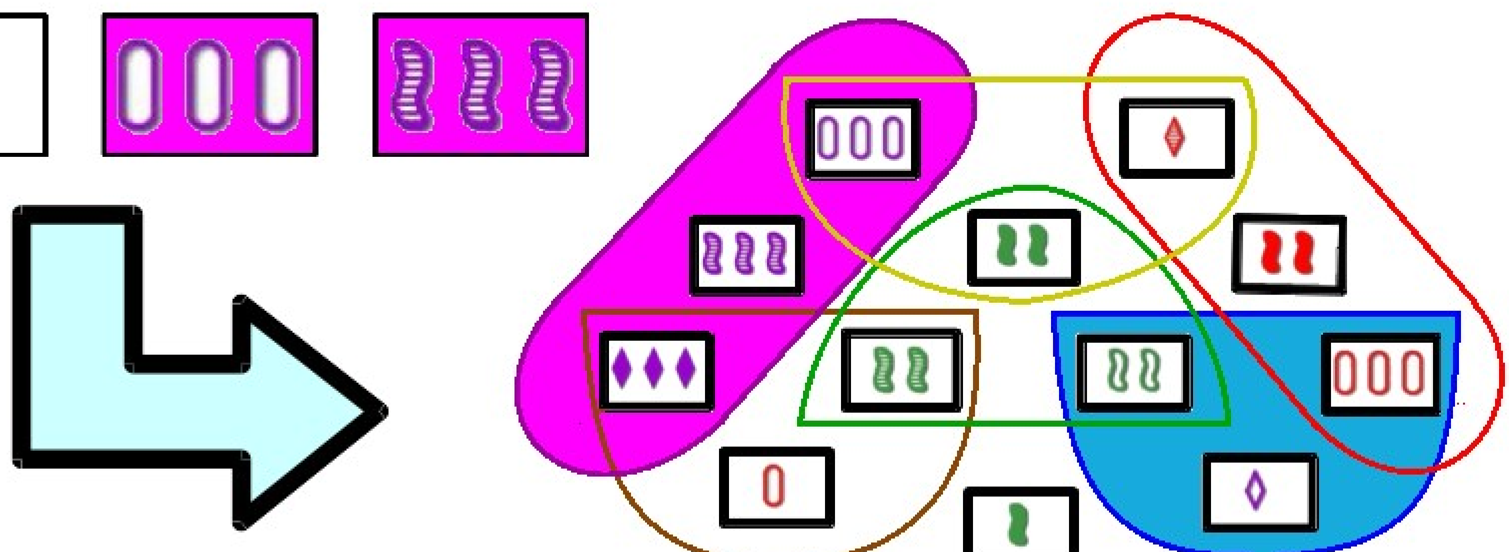


Restriction of Independent Edge Dominating Set to SET hypergraphs.



Multi-round variations as hypergraph problems

<u>Our results:</u>

- *Both restrictions remain NP-hard.*
- *Independent Edge Dominating Set on general 3-uniform hypergraphs is FPT*

  →*min 3-rSET parameterized by r is FPT*

([Fellows et al 2008] 3-*Set Packing parameterized by size of solution is FPT).*

SET Summary

In connection to SET:

- SET can be reformulated as Perfect Multidimensional Matching, Set Packing, Edge Dominating Set.
- Complexity-wise:
  - One-round SET parameterized by #values is W[1]-hard.
  - Multi-round max & min 3-SET are NP-hard and FPT parameterized by #rounds.

Beyond SET:

- Perfect multidimensional matching parameterized by the size of the dimensions is W[1]-hard.
- Independent edge dominating set parameterized by the size of the dominating set is FPT on 3-uniform h-graphs.

The Game of

Joint with Michael Lampis, Kazuhisa Makino, and Yushi Uno

Solitaire UNO - Rules

Deck of cards:

- c colors;
- b ranks.

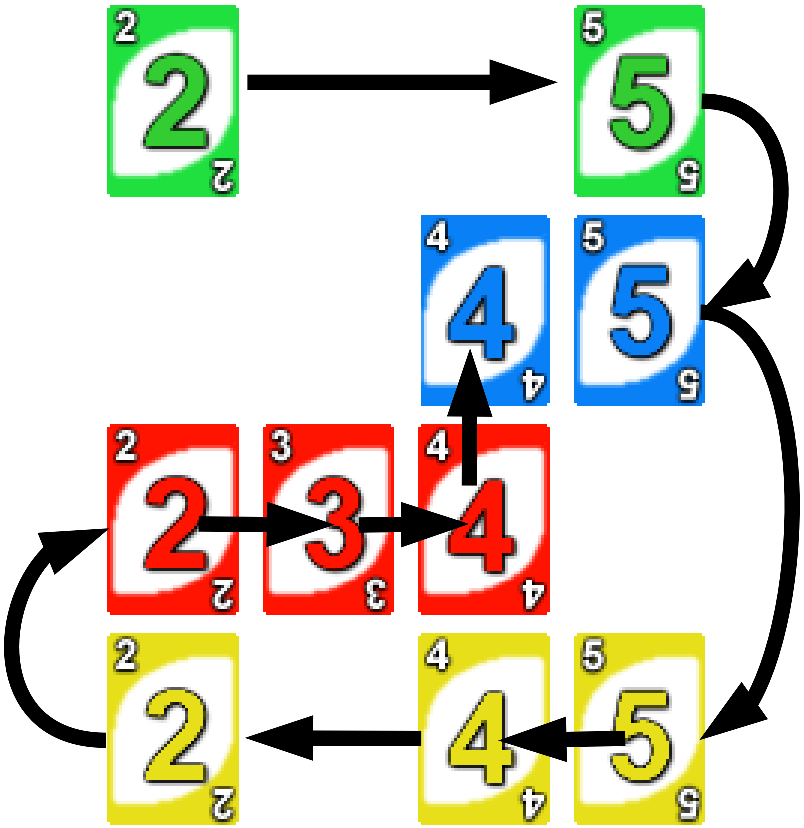
- Given m cards, discard them one by one following the matching rule.

  Matching rule: cards agree either in color or in rank.

Solitaire UNO - Rules

Deck of cards:

- c colors;
- b ranks.



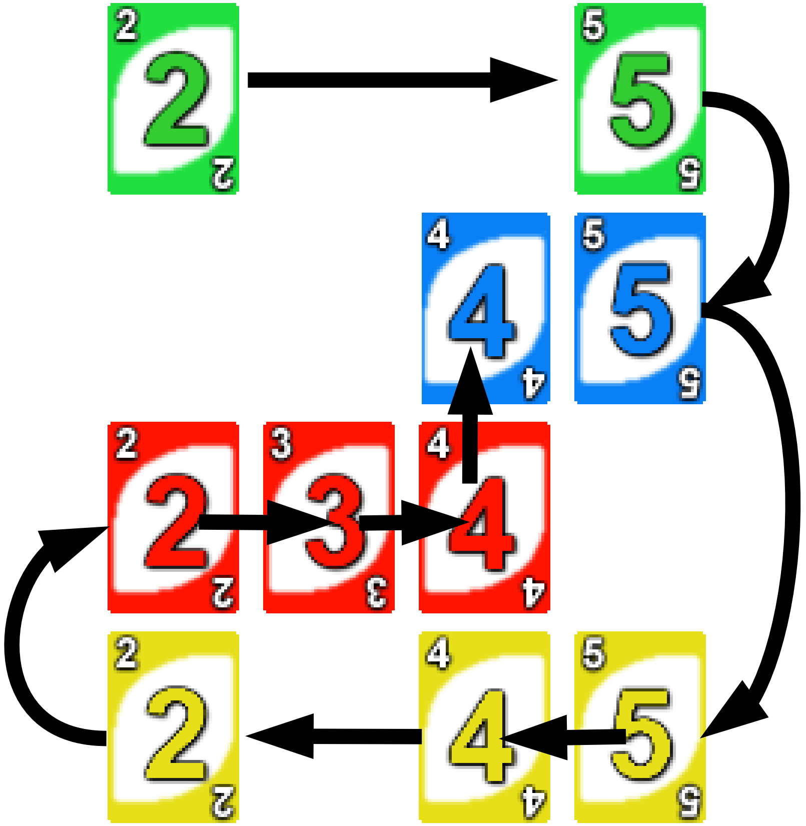
- Given m cards, discard them one by one following the matching rule.

  Matching rule: cards agree either in color or in rank.

Solution: 2, 5, 5, 5, 4, 2, 2, 3, 4, 4

Solitaire UNO as Hamiltonicity

- Given an m-vertex graph, find a permutation of the vertices such that consecutive vertices are neighbors.



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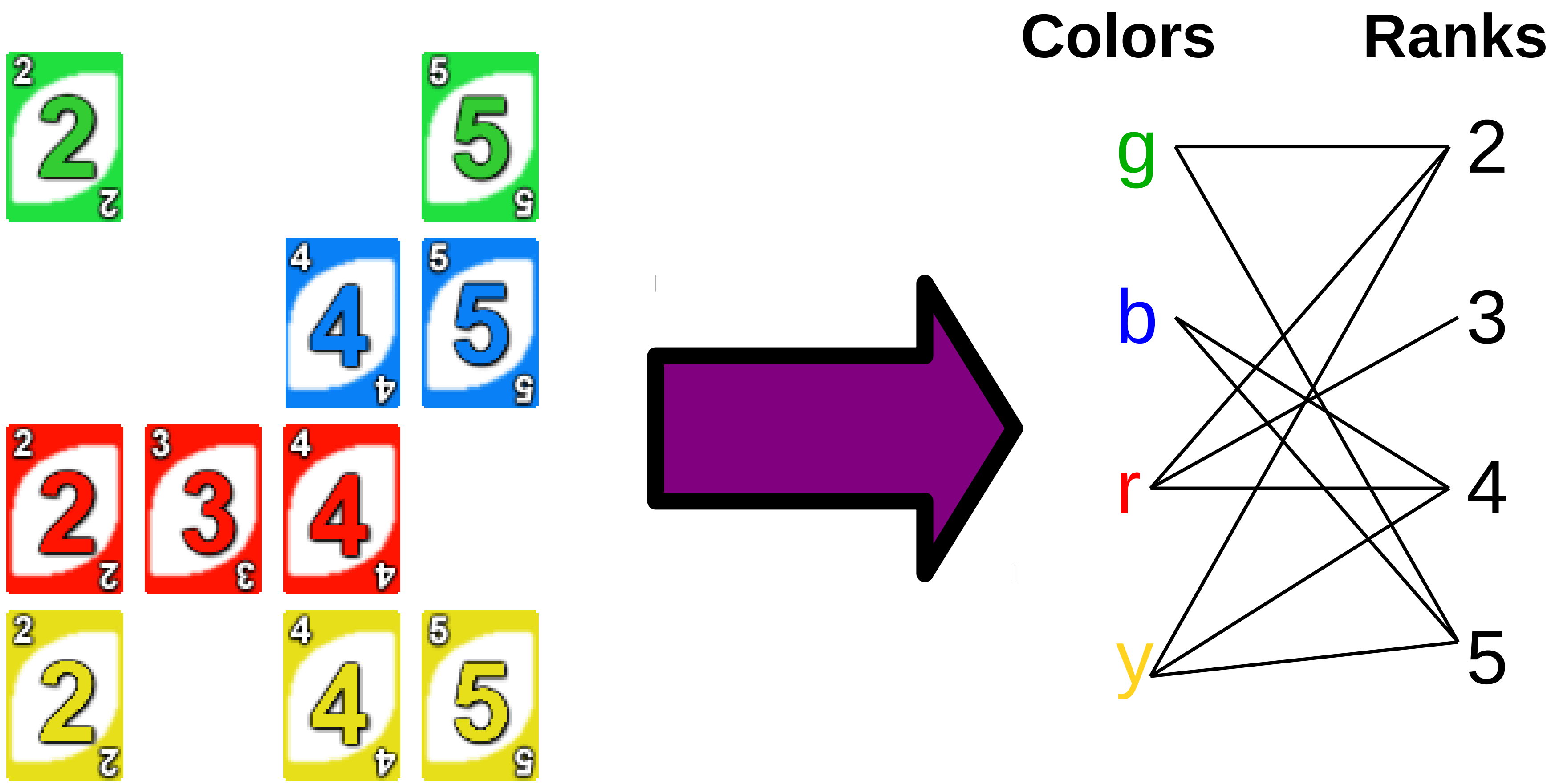
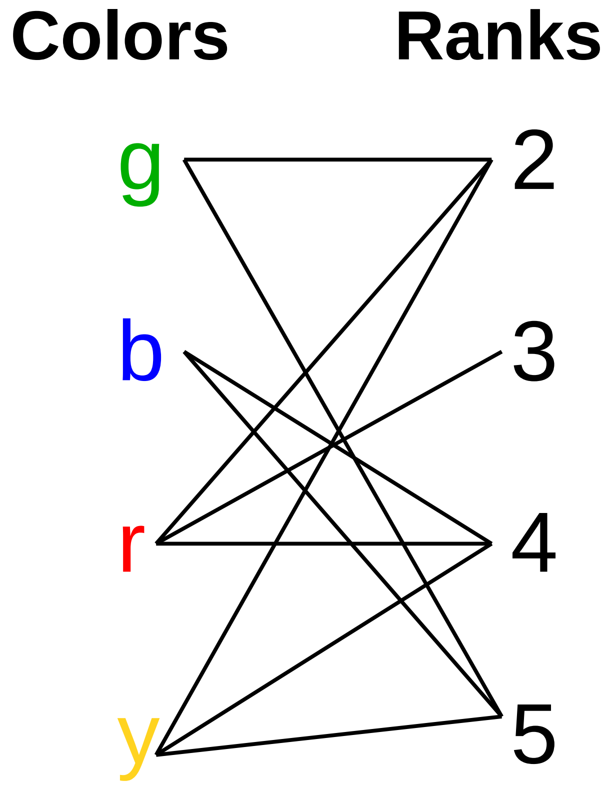
Solitaire UNO as Edge-Hamiltonicity



Solitaire UNO as Edge-Hamiltonicity

Edge-Hamiltonian Path:

"Ordering of the edges such that consecutive edges share a common attribute."



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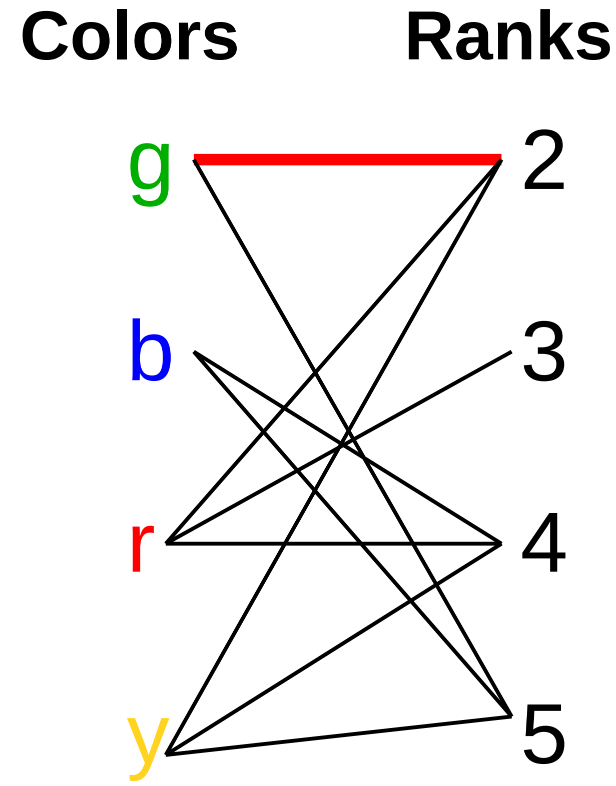
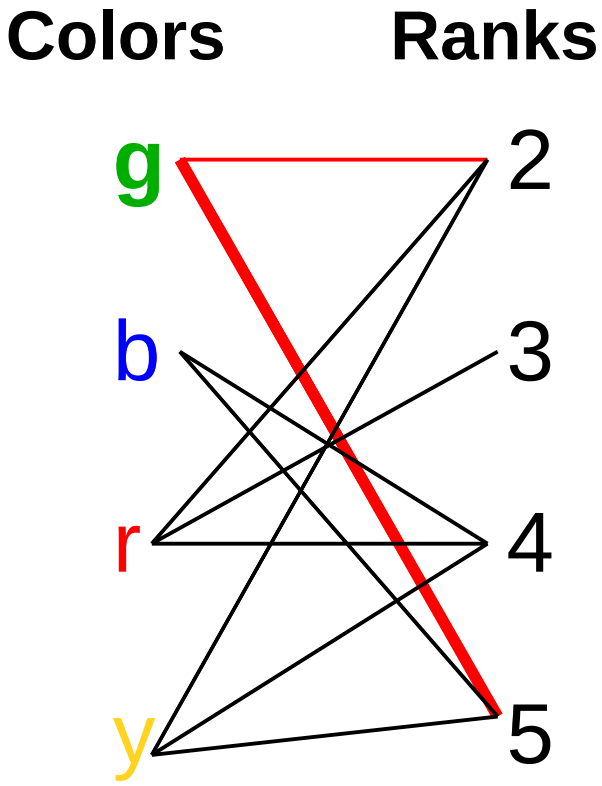


2,

Solitaire UNO as Edge-Hamiltonicity



Edge-Hamiltonian Path:

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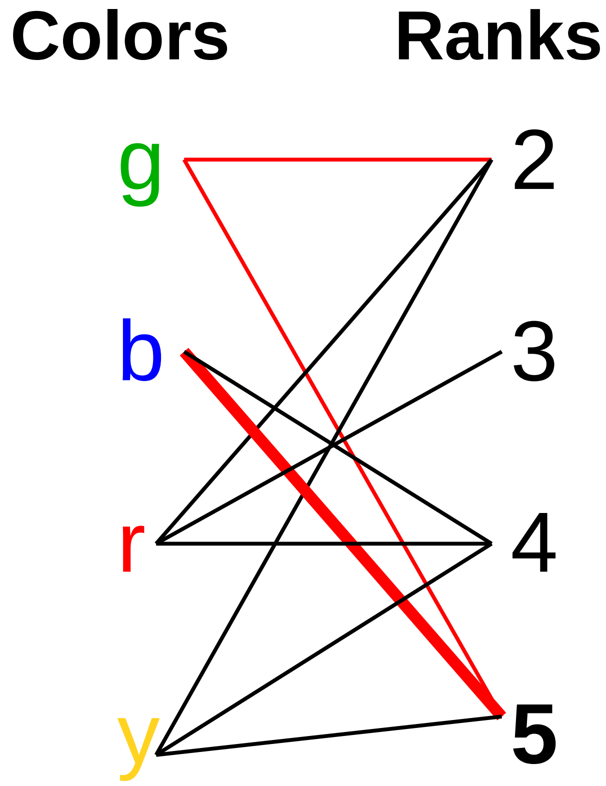
2, 5,

Solitaire UNO as Edge-Hamiltonicity



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2, 5, 5,

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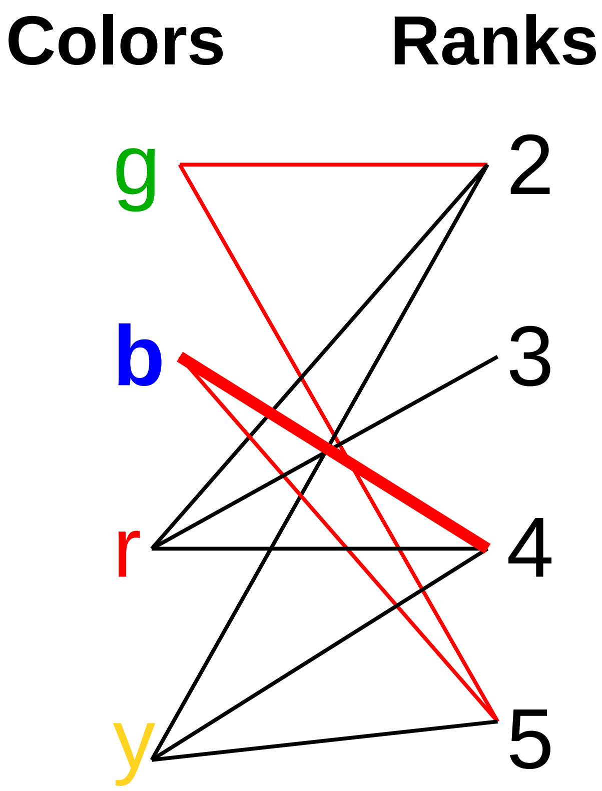
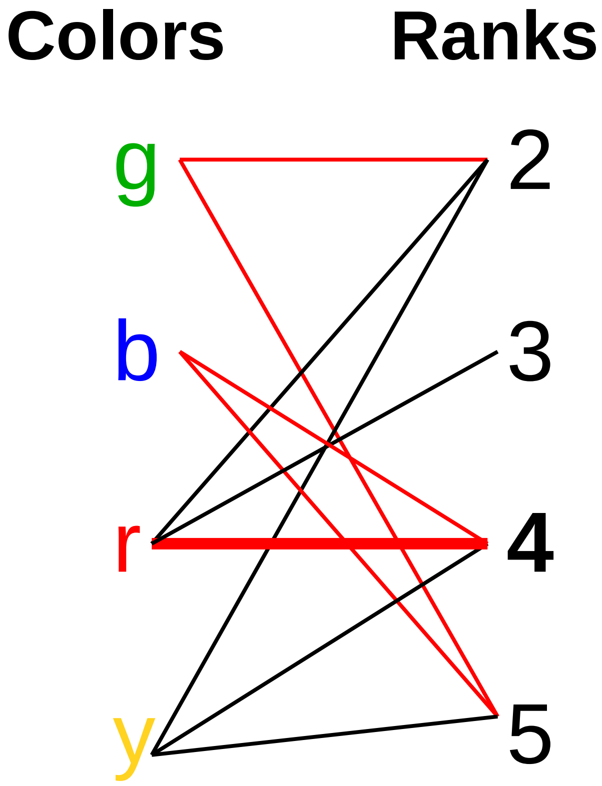


Edge-Hamiltonian Path:

"Ordering of the edges such that consecutive edges share a common attribute."

2, 5, 5, 4,

Solitaire UNO as Edge-Hamiltonicity



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2, 5, 5, 4, 4,

Solitaire UNO as Edge-Hamiltonicity

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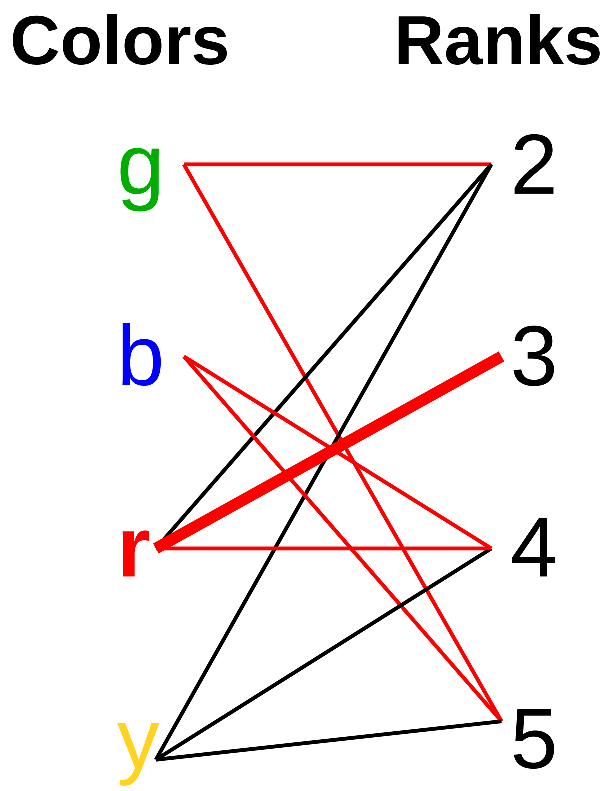

2, 5, 5, 4, 4, 3,

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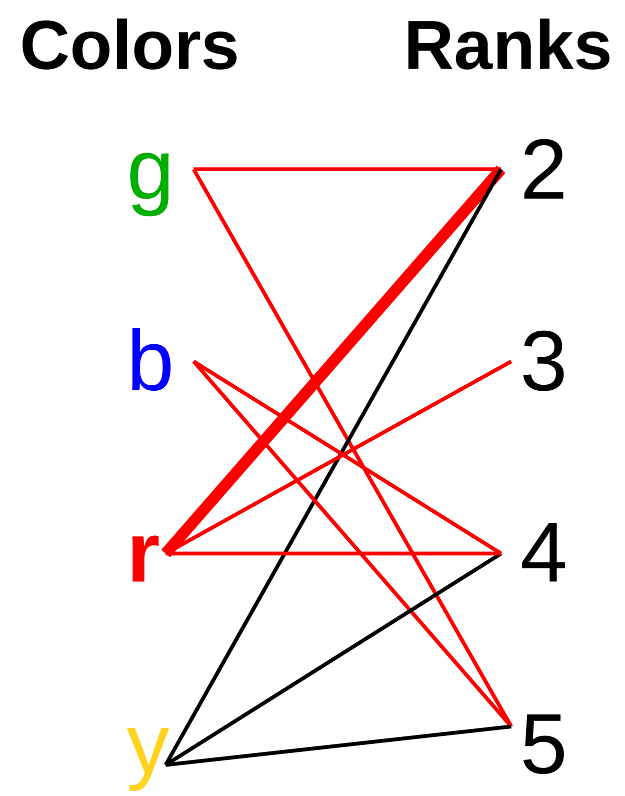

2, 5, 5, 4, 4, 3, 2,

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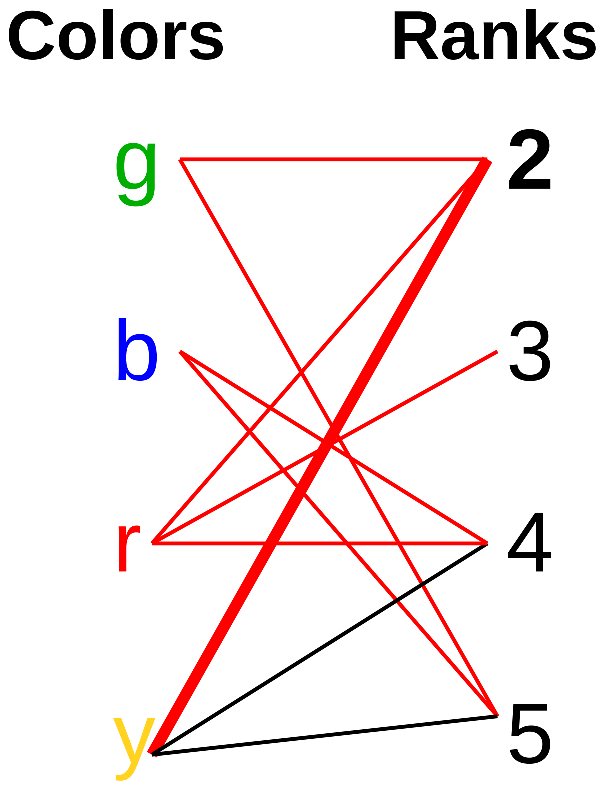

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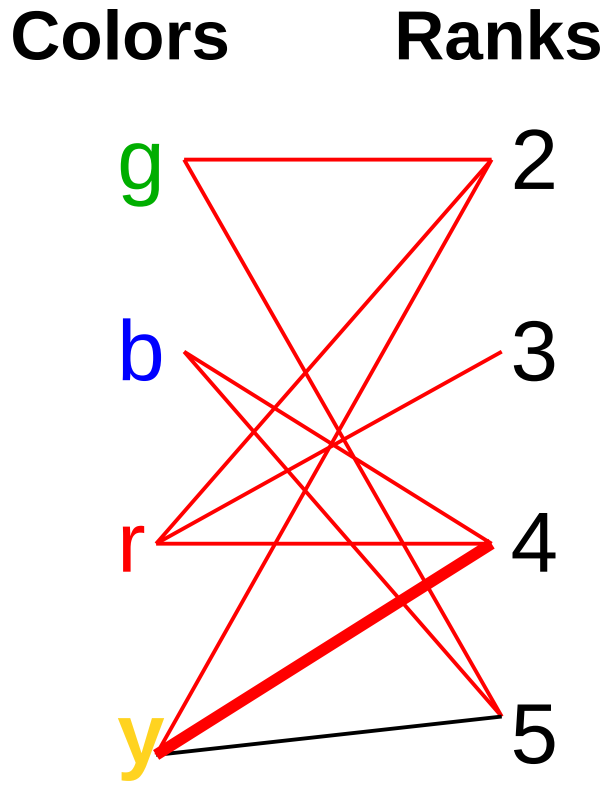

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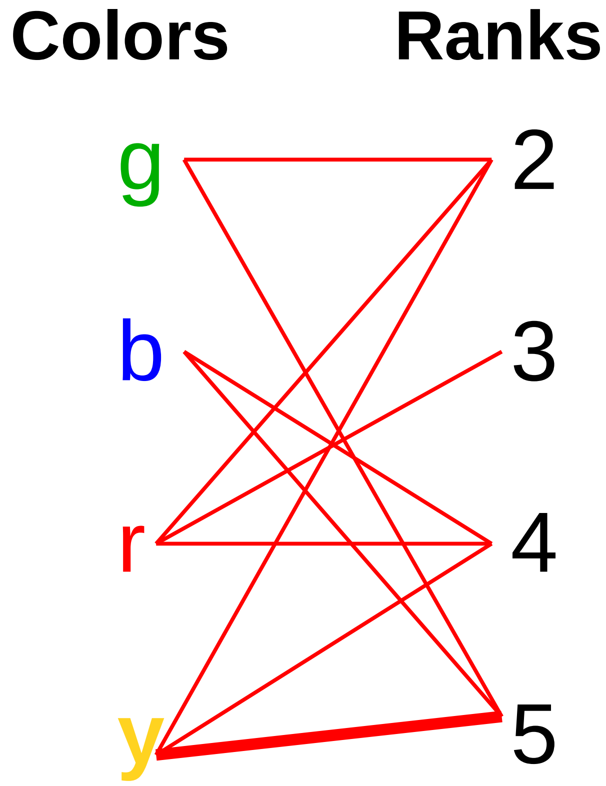
Edge-Hamiltonian Path:

"Ordering of the edges such that consecutive edges share a common attribute."



2, 5, 5, 4, 4, 3, 2, 2, 4, 5

Solitaire UNO – Previous Results

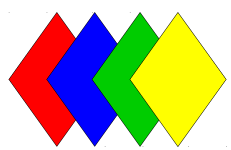
- [Bertossi 1981]: Edge-Hamiltonian path is NP-complete.
- [Lai, Wei 1993]: Edge-Hamiltonian path is NP-complete on bipartite graphs.

→ [Demaine et al 2014]: Solitaire UNO is NP-complete.

Parameterized Solitaire UNO

In the original game, the number of colors *c* is quite smaller than the number of ranks *b.*

c



b

1 2 3 4 5 6 7 8 9 0

Can we do better under the assumption that c<<b?

→ *Study parameterized complexity!*

Parameterized Results

[Demaine et al 2014]: Solitaire UNO with 2 attributes (color & rank) can be solved in b^{O(c^2)} time.

→ XP #colors c is a parameter.

Our results:

- Solitaire UNO with unbounded attributes r is FPT;
- When r = 2, it even admits a cubic kernel.
- EHP is FPT parameterized by the size of a given vertex cover (or hitting set in case of hypergraphs)

Solitaire UNO is FPT (sketch)

From an EHP **SOL**, we can construct an EHP **SOL'** where each color-group appears at most c times.

SOL: 1 5 5 3 ... 2 8 8 7

Solitaire UNO is FPT (sketch)

From an EHP **SOL**, we can construct an EHP **SOL'** where each color-group appears at most c times.

SOL': 1 5 8 2 ... 3 5 8 7

Solitaire UNO is FPT (sketch)

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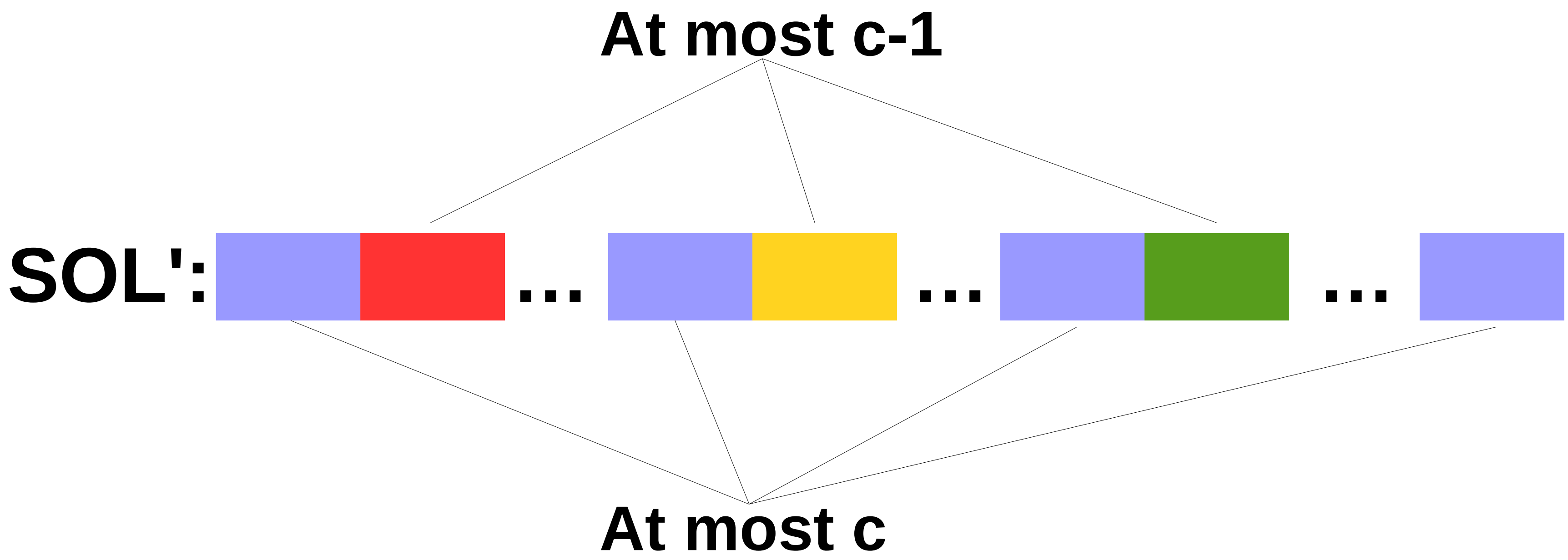
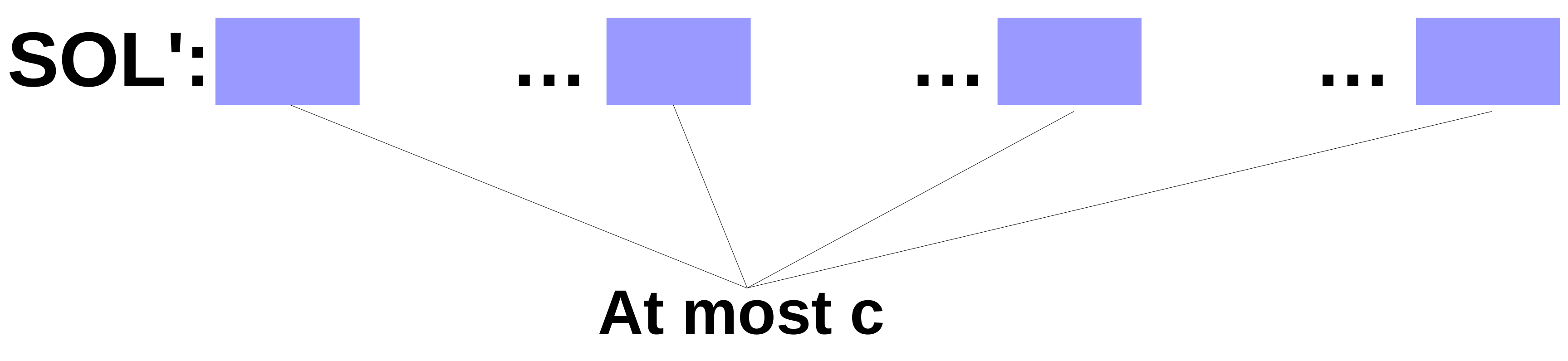
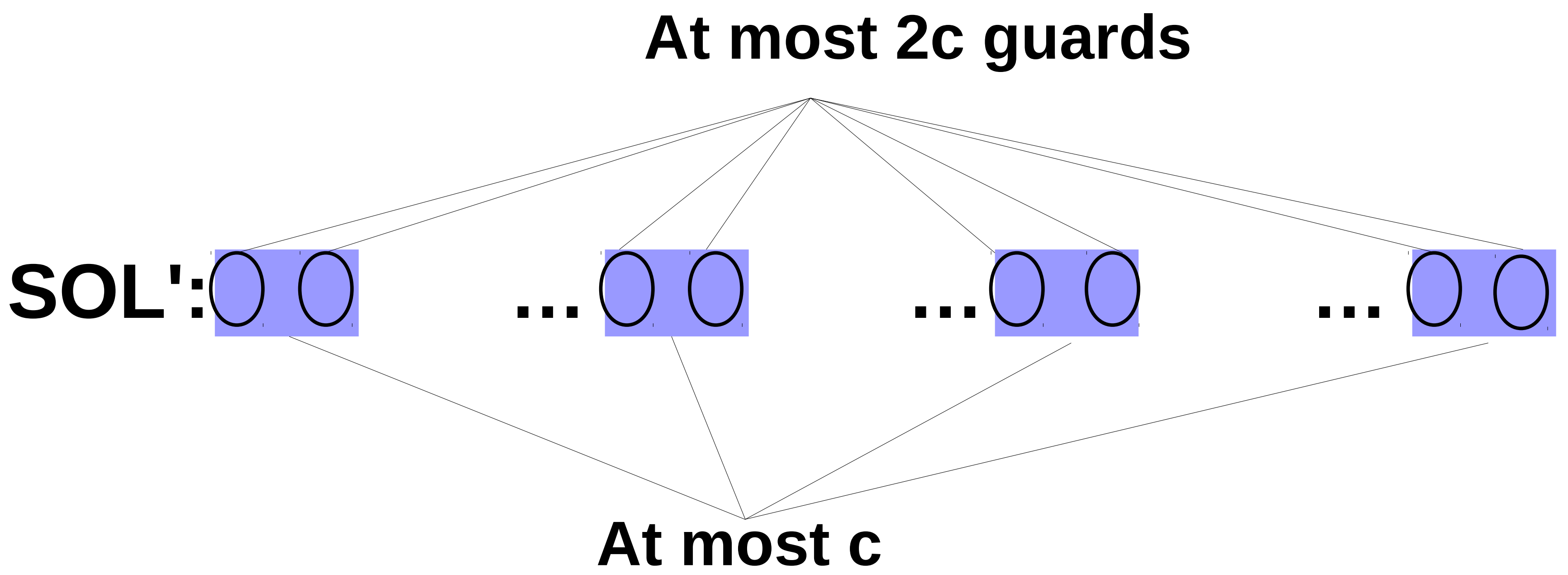


Solitaire UNO is FPT (sketch)

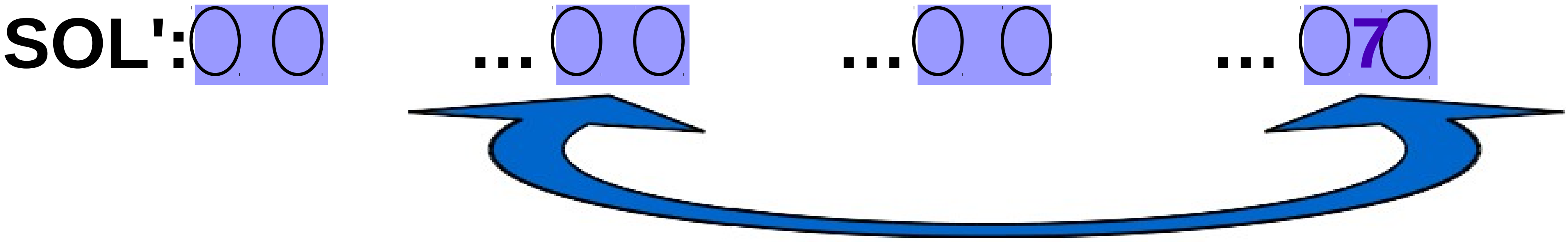


Solitaire UNO is FPT (sketch)



Solitaire UNO is FPT (sketch)

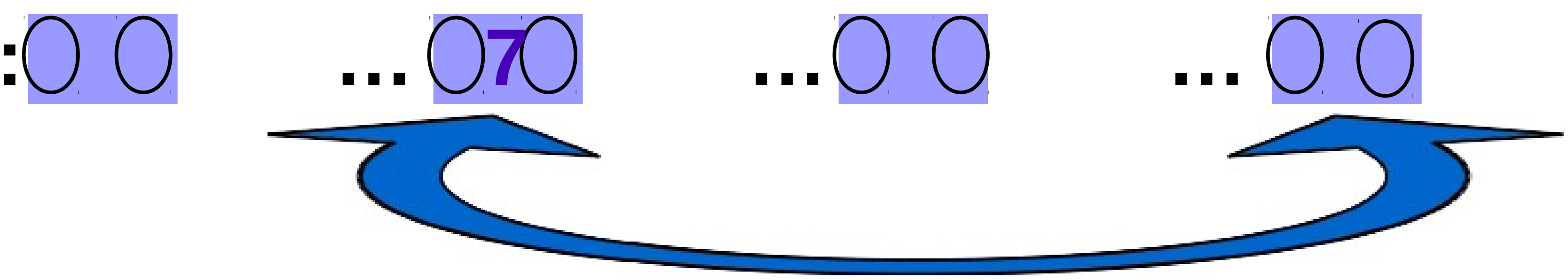
- All other cards can go in-and-out freely.



Solitaire UNO is FPT (sketch)

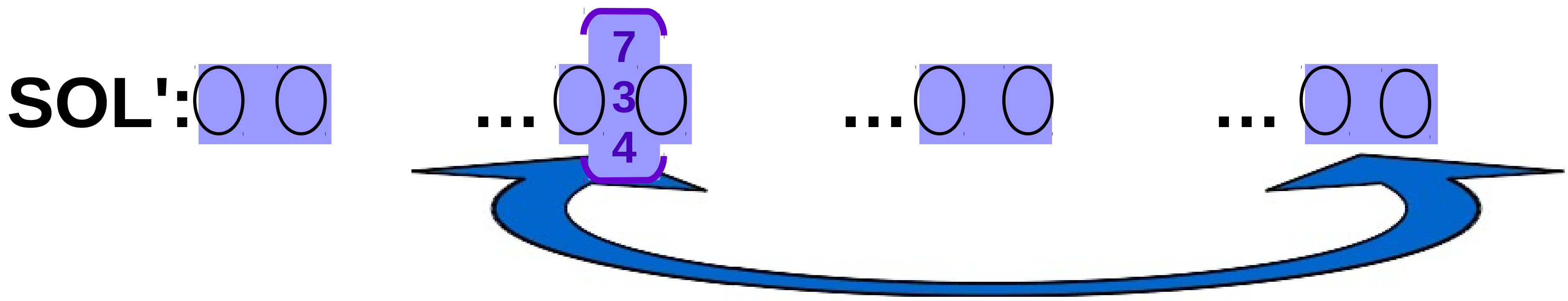
- All other cards can go in-and-out freely.

SOL':



Solitaire UNO is FPT (sketch)

- All other cards can go in-and-out freely.
- Argument works even for unbounded attributes.



Solitaire UNO is FPT (sketch)

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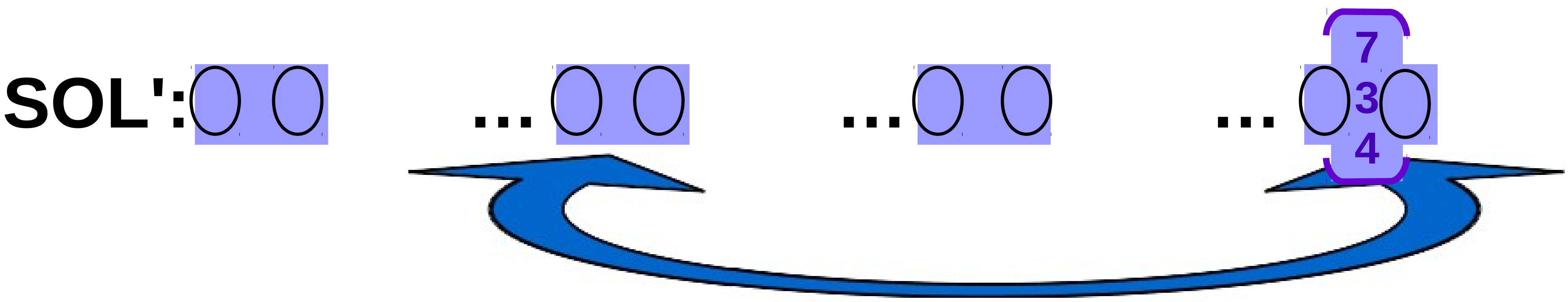
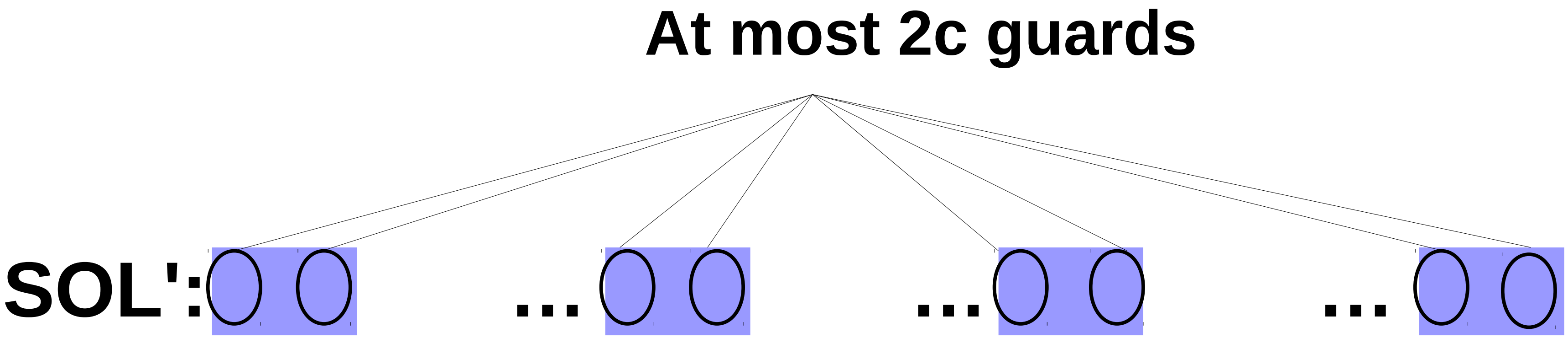


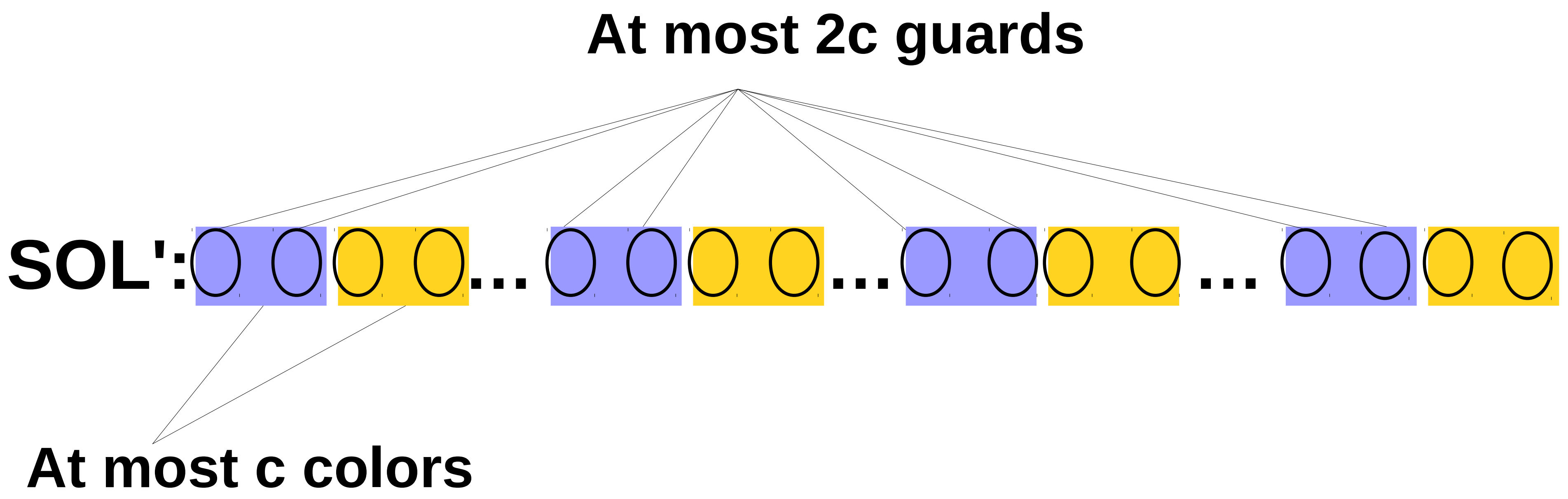
Solitaire UNO is FPT (sketch)



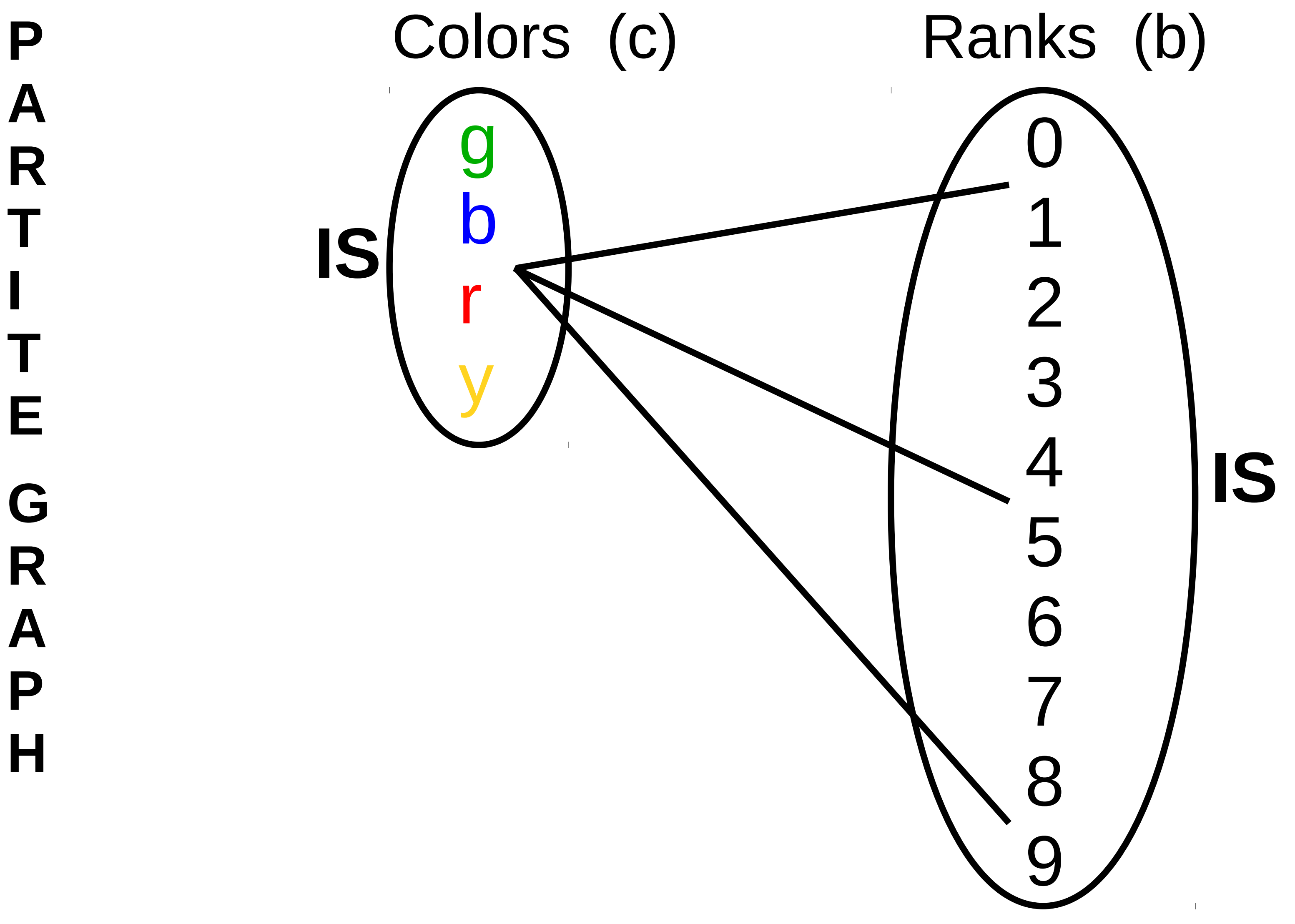
Solitaire UNO is FPT (sketch)

- **SOL'** has at most $2c^2$ guards

  (backbone of the solution)



EHP on Graphs of small Vertex Cover



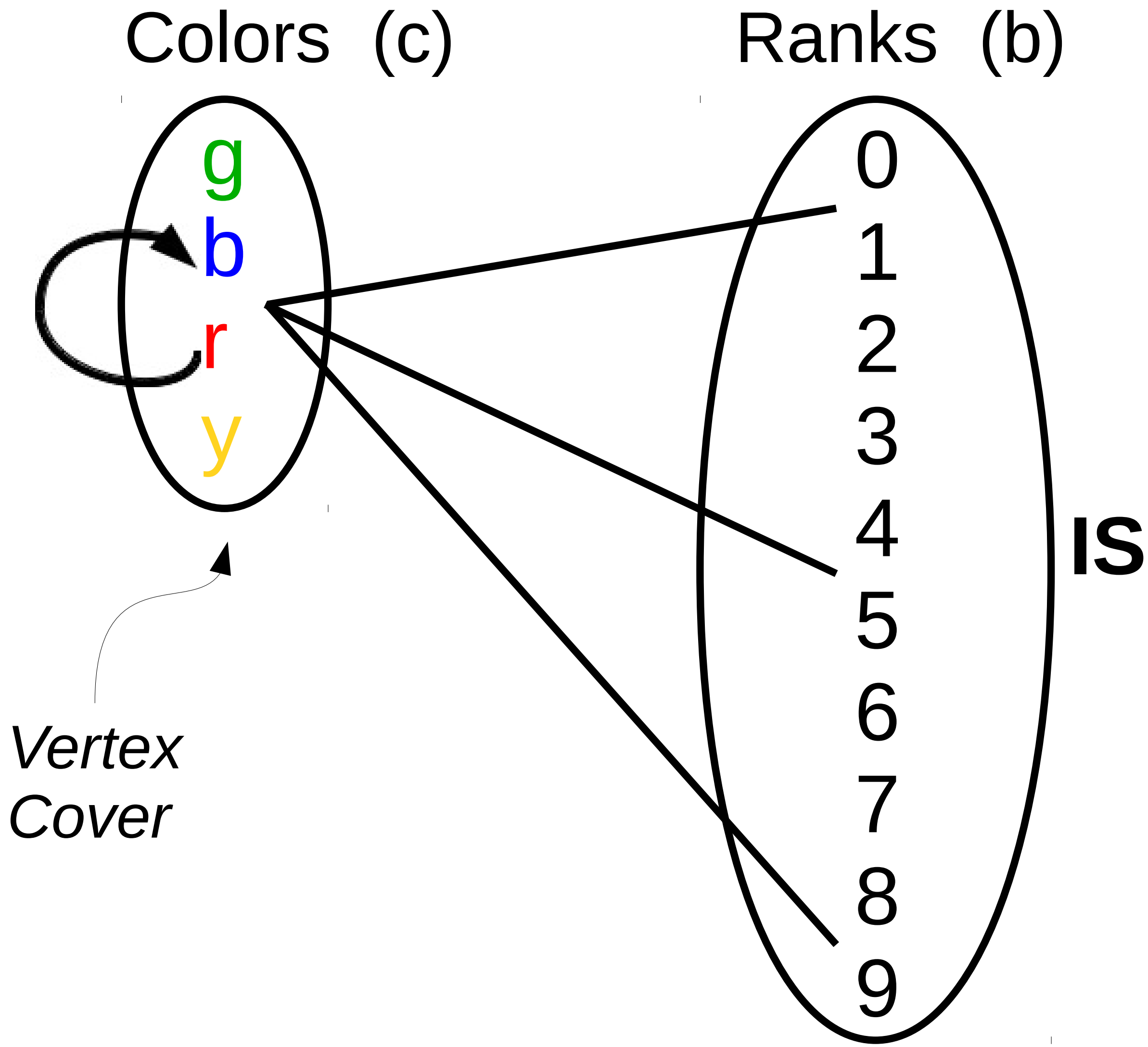
EHP on Graphs of small Vertex Cover

G
R
A
P
H

O
F

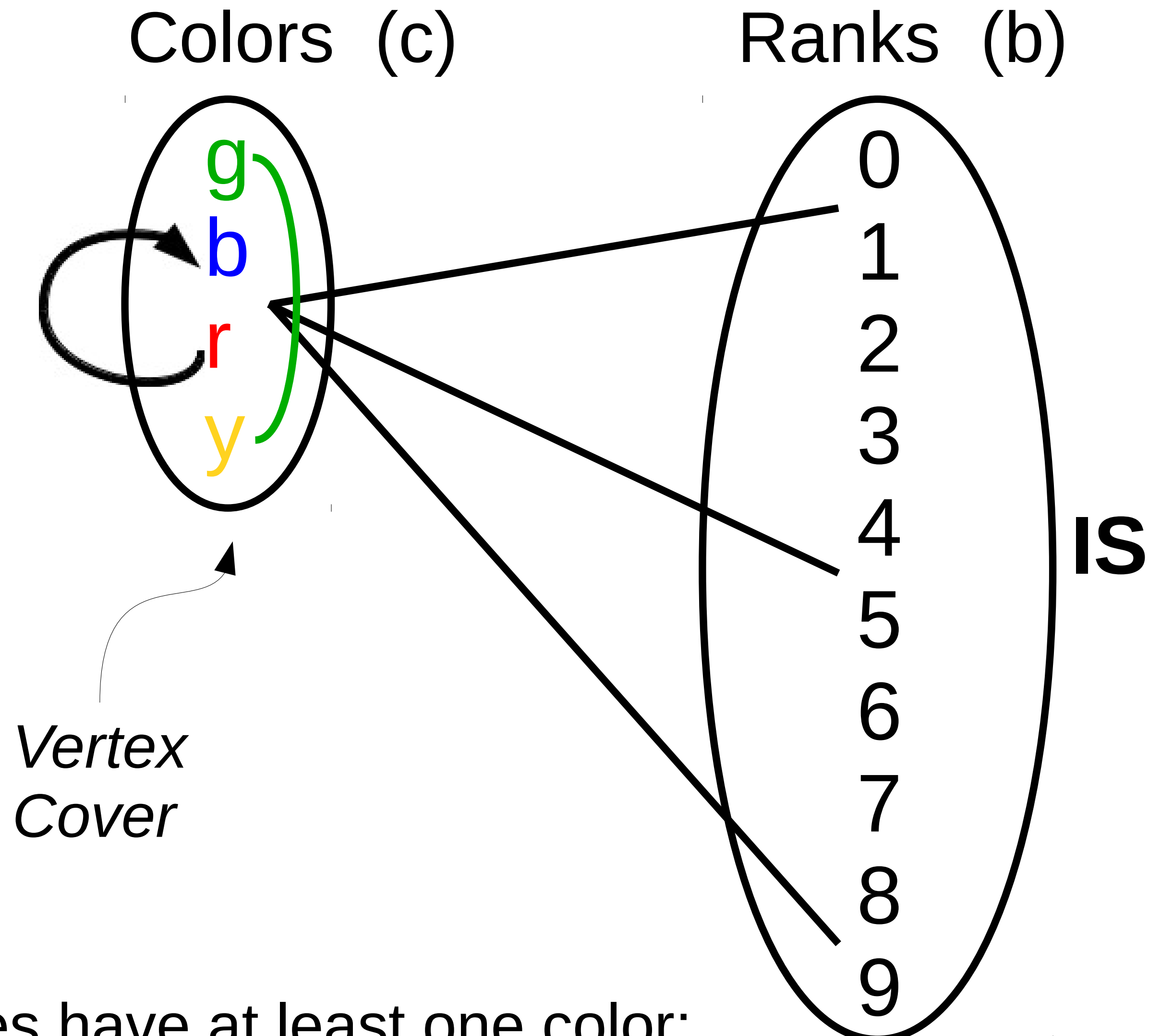
S
M
A
L
L

V
C



EHP on Graphs of small Vertex Cover

G R A P H O F S M A L L V C



All edges have at least one color;
pick one representing color.

$$\left(\begin{array}{c}\text{Parenthesis:}\\ \text{EHP parameterized by tw and cw}\end{array}\right)$$

Treewidth and Cliquewidth

- Structural graph parameters:
  - Treewidth measures how tree-like a graph is.
  - Cliquewidth again measures graph complexity but is more general than tw (graphs of bounded tw also have bounded cw).

Treewidth and Cliquewidth

- Problems expressible in MSO_1 are FPT on graphs of bounded tw, cw.
- tw is algorithmically more tractable than cw:

  – Problems expressible in MSO_2 logic are tractable for graphs of bounded tw but not always for graphs of bounded cw, ex. (vertex) Hamiltonicity.

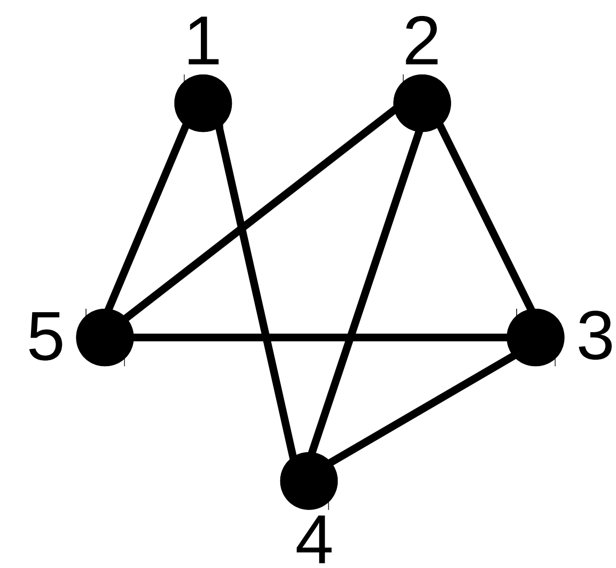
  <u>*Question:*</u> *Can EHP be expressed in some MSO?*

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>Dominating Eulerian Subgraph</u>

“Find a connected subgraph G' of G, st:

1. *G' is Eulerian;*
2. *All remaining edges of G are covered by a vertex in G'.”*



Harary & Nash-Williams (1965): Edge-Hamiltonian Path is equivalent with Dominating Eulerian Subgraph.

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

__Dominating Eulerian Subgraph__

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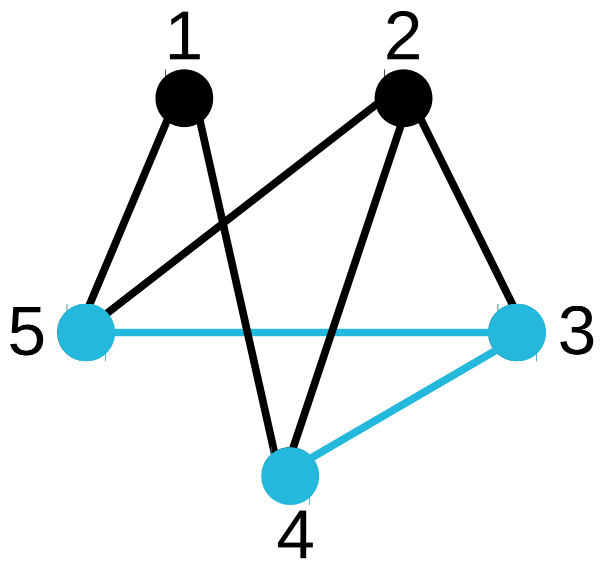

Eulerian path: 53, 34

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>*Dominating Eulerian Subgraph*</u>

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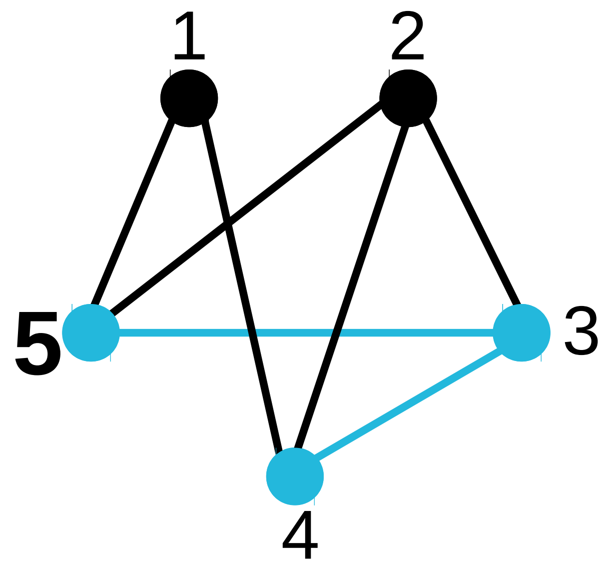

Eulerian path: 53, 34
Edge-Hamiltonian Path:

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>Dominating Eulerian Subgraph</u>

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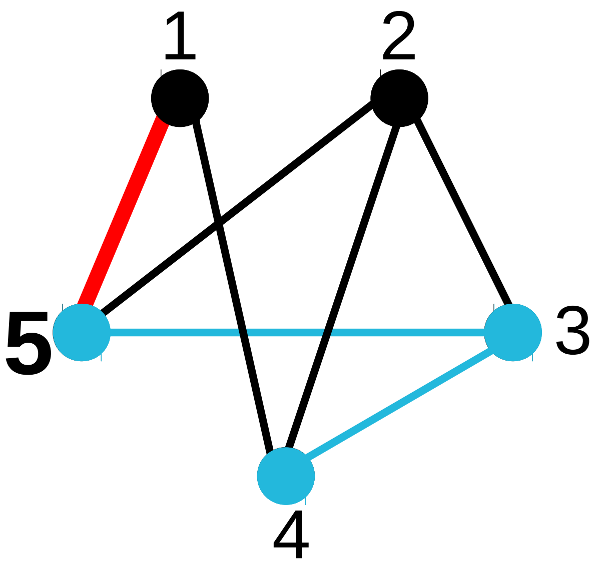

Eulerian path: 53, 34
Edge-Hamiltonian Path: 51

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>Dominating Eulerian Subgraph</u>

“Find a connected subgraph G' of G, st:

1. *G' is Eulerian;*
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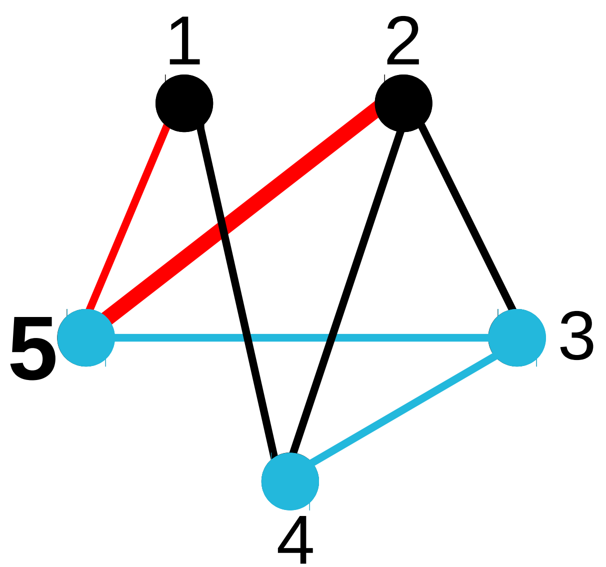

Eulerian path: 53, 34
Edge-Hamiltonian Path: 51, 52

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>Dominating Eulerian Subgraph</u>

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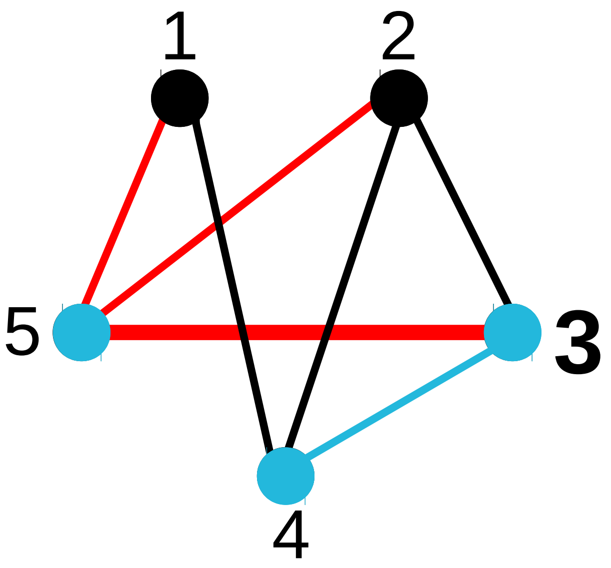

Eulerian path: 53, 34
Edge-Hamiltonian Path: 51, 52, 53

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>*Dominating Eulerian Subgraph*</u>

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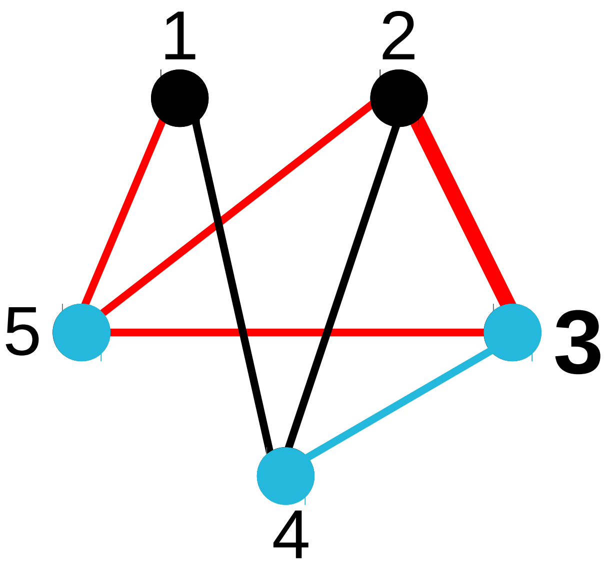

Eulerian path: 53, 34

Edge-Hamiltonian Path: 51, 52, 53, 32

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>Dominating Eulerian Subgraph</u>

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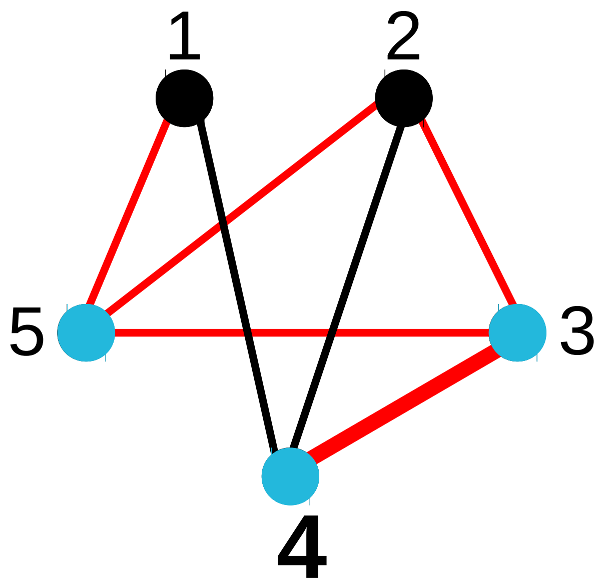

Eulerian path: 53, 34
Edge-Hamiltonian Path: 51, 52, 53, 32, 34

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>Dominating Eulerian Subgraph</u>

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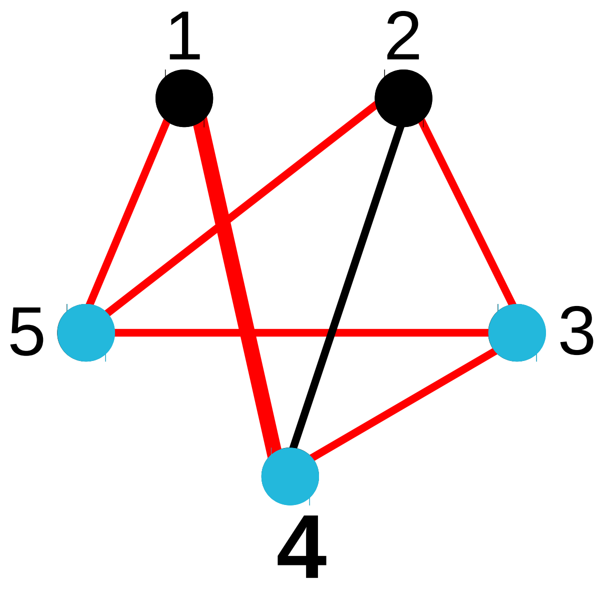

Eulerian path: 53, 34

Edge-Hamiltonian Path: 51, 52, 53, 32, 34, 41

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>Dominating Eulerian Subgraph</u>

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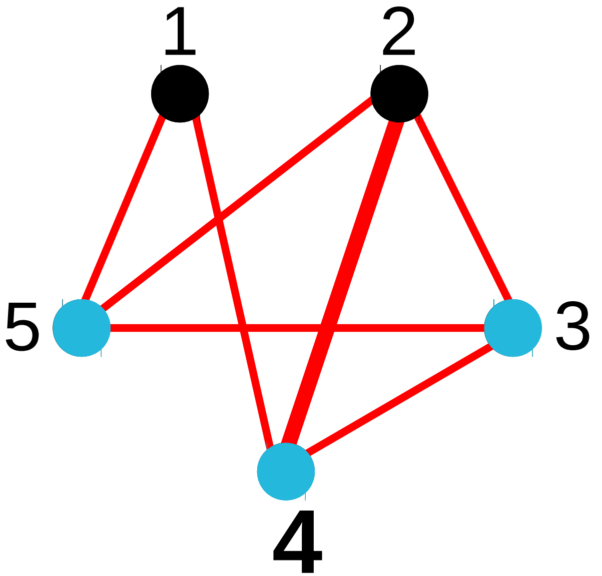

Eulerian path: 53, 34
Edge-Hamiltonian Path: 51, 52, 53, 32, 34, 41, 42

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

<u>Dominating Eulerian Subgraph</u>

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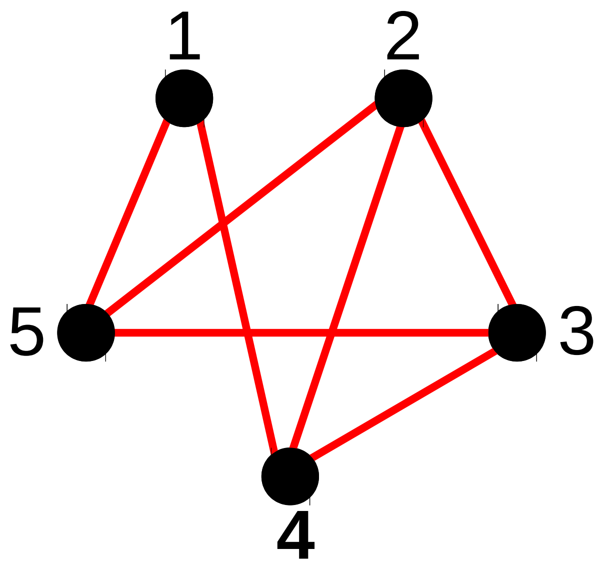

All the above properties are expressible in $CMSO_2$.

➡ *Dominating Eulerian Subgraph (thus EHP) parameterized by tw is FPT.*

Edge-Hamiltonian Path as Dominating Eulerian Subgraph

Dominating Eulerian Subgraph

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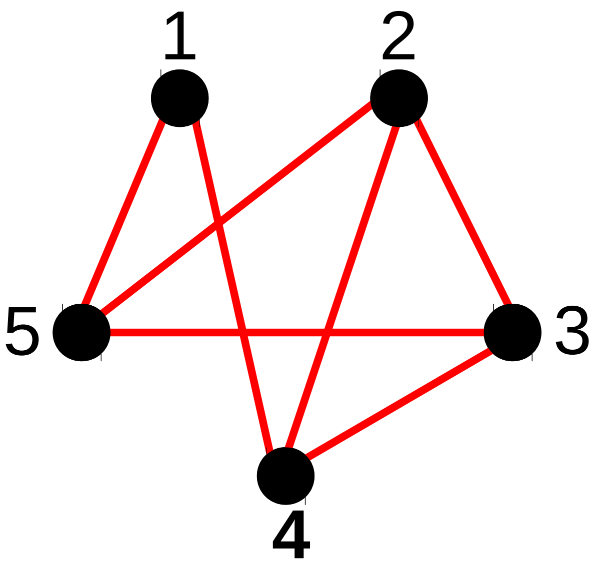
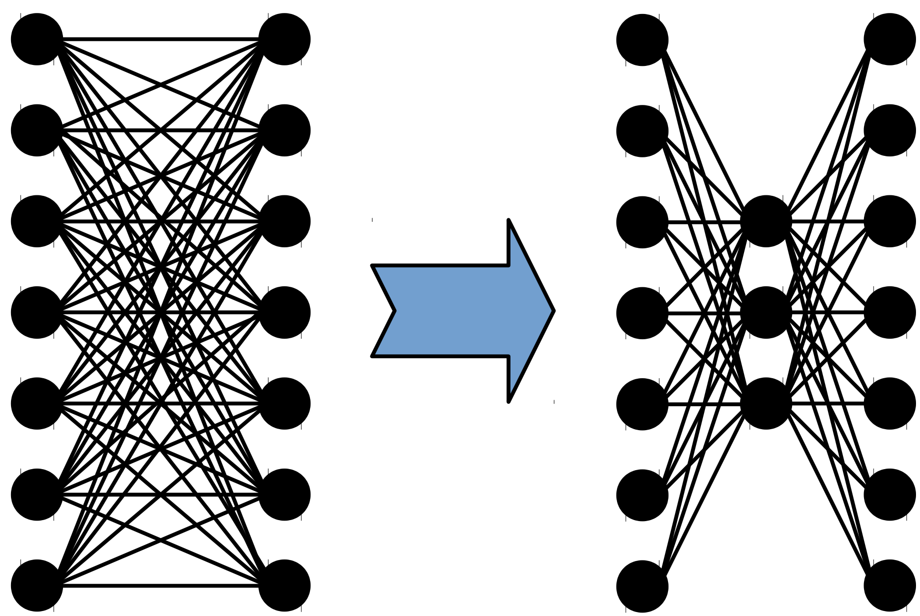


All the above properties are expressible in $CMSO_2$.

- *Dominating Eulerian Subgraph (thus EHP) parameterized by tw is FPT.*
- *EHP parameterized by cw is FPT.*

EHP parameterized by cw is FPT

Theorem (Gursky & Wanke 2000): If G has cw k and does not contain $K_{t,t}$ as a subgraph, then G has tw at most 3kt.



(Closing Parenthesis)

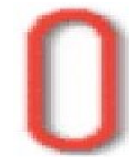
UNO Summary

In connection to UNO:

- UNO can be reformulated as (Edge) Hamiltonian Path.
- Complexity-wise:
  - UNO parameterized by the size of one of the card's attributes is FPT.
  - For 2 attributes, it admits a cubic kernel.

Beyond UNO:

- EHP parameterized by |VC| admits a cubic kernel.
- EHP on hypergraphs parameterized by |HS| is FPT.
- EHP parameterized by tw & cw is FPT.

T  N K Y  U !

I) Michael Lampis and Valia Mitsou: T**he Computational Complexity of the Game of Set and its Theoretical Applications.** *LATIN 2014.*

II) Michael Lampis, Kazuhisa Makino, Valia Mitsou, and Yushi Uno: **Parameterized Edge Hamiltonicity.** *WG 2014.*